Strength of polynomials via polynomial functors

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The strength of polynomials

Let f be a homogeneous polynomial of degree $d \geq 2$.

Definition

The *strength* of f is the minimal number $\operatorname{str}(f) := r \geq 0$ such that

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

with $g_1, h_1, \ldots, g_r, h_r$ homogeneous polynomials of degree $\leq d - 1$.

Defined by Ananyan and Hochster in order to prove Stillman's Conjecture. Used by Erman, Sam and Snowden in their work on big polynomial rings. Plays a big role when studying the geometry of polynomial functors. Has also been defined for sections of line bundles over algebraic varieties by Ballico and Ventura.

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Theorem (Ballico-B-Oneto-Ventura)

The set

$$\{f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \mathrm{str}(f) \leq 3\}$$

is not Zariski-closed for $n \gg 0$.

The strength of polynomials

Example ($d = 2$)

Let

$$f = (x_1, \dots, x_n) \cdot A \cdot (x_1, \dots, x_n)^\top, \quad A \in \mathbb{C}^{n \times n} \text{ with } A^\top = A$$

be a homogeneous polynomial of degree 2. By applying a coordinate transformation (or replacing A be a congruent matrix), we may assume that $A = \mathrm{Diag}(\mathbf{1}_k, \mathbf{0}_{n-k})$ and $f = x_1^2 + \ldots + x_k^2$.

If $f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$ with

$$g_j = (x_1, \dots, x_n) \cdot v_j^\top \text{ and } h_j = w_j \cdot (x_1, \dots, x_n)^\top,$$

then $A = (v_1^\top w_1 + w_1^\top v_1) + \ldots + (v_r^\top w_r + w_r^\top v_r)$. So $k \leq 2r$.

As $x_j^2 + x_{j+1}^2 = (x_j + i x_{j+1})(x_j - i x_{j+1})$, we have $\mathrm{str}(f) = \lceil k/2 \rceil$.

The strength of polynomials

Definition

The *slice rank* of f is the minimal number $\mathrm{slrk}(f) := r \geq 0$ such that
$$f = g_1 \cdot \ell_1 + \ldots + g_r \cdot \ell_r$$
with $g_1, \ldots, g_r$ of degree $d-1$ and $\ell_1, \ldots, \ell_r$ linear.

Proposition (Tao-Sawin, Derksen-Eggermont-Snowden)

The set
$$\{f \in \mathbb{C}[x_1, \ldots, x_n]_d \mid \mathrm{slrk}(f) \leq k\}$$
is Zariski-closed for all $d \geq 2$, $n \geq 1$ and $k \geq 0$.

Proof.

It is the cone of the projection of
$$\{([f], V) \in \mathbb{P}(\mathbb{C}[x_1, \ldots, x_n]_d) \times \mathbb{Gr}(n-k, n)\} \mid f(V) = 0\}$$

The strength of polynomials

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Theorem

For $d \geq 3$ and $n \geq 1$, the generic slice rank in $\mathbb{C}[x_1, \ldots, x_n]_d$ is

$$\mathrm{slrk}^{\circ}_{d,n} := \min\left\{ r \in \mathbb{Z} \,\middle|\, r(n-r) \geq \binom{d-r+n-1}{d} \right\}.$$

The strength of polynomials

Conjecture

The generic strength and generic slice rank coincide.

Example (Fermat polynomials)

Take $f = x_1^d + \ldots + x_n^d$ with $d \geq 2$.

As $x_j^d + x_{j+1}^d$ is reducible, we have $\mathrm{str}(f) \leq \lceil n/2 \rceil$.

Ananyan-Hochster Trick:
If $f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$, then

$$\mathrm{Sing}\{f = 0\} = \{0\}$$

contains the variety defined by $g_1, h_1, \ldots, g_r, h_r$ and hence has codimension $\leq 2r$. So we find $\mathrm{str}(f) \geq \lceil n/2 \rceil$.

The strength of polynomials

Theorem (Ballico-B-Oneto-Ventura)

The set

$$\{f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \mathrm{str}(f) \leq 3\}$$

is not Zariski-closed for $n \gg 0$.

Question

Is the set

$$\{f \in \mathbb{C}[x_1, \ldots, x_n]_d \mid \mathrm{str}(f) \leq 2\}$$

Zariski-closed for all $d \geq 2$ and $n \geq 1$?

Proof is non-constructive and uses polynomial functors.

Polynomial functors: Definition

Let Vec be the category of finite-dimensional vector spaces.

Definition

A functor $P\colon \mathrm{Vec} \to \mathrm{Vec}$ sends

$$\begin{aligned} V &\mapsto P(V) \\ (\ell\colon V \to W) &\mapsto (P(\ell)\colon P(V) \to P(W)) \end{aligned}$$

such that $P(\mathrm{id}_V) = \mathrm{id}_{P(V)}$ and $P(\ell_1 \circ \ell_2) = P(\ell_1) \circ P(\ell_2)$.

Examples

Take $U \in \mathrm{Vec}$ fixed.

- $C_U\colon V \mapsto U, \ell \mapsto \mathrm{id}_U$
- $T\colon V \mapsto V, \ell \mapsto \ell$

Polynomial functors: Definition

You can add and multiply two functors $P, Q\colon \mathrm{Vec} \to \mathrm{Vec}$.

$$\begin{array}{rclrcl} (P \oplus Q)(V) & = & P(V) \oplus Q(V), & (P \otimes Q)(V) & = & P(V) \otimes Q(V) \\ (P \oplus Q)(\ell) & = & P(\ell) \oplus Q(\ell), & (P \otimes Q)(\ell) & = & P(\ell) \otimes Q(\ell) \end{array}$$

You can take subfunctors and quotients:
We have $Q \subseteq P$ when $Q(V) \subseteq P(V)$ and $P(\ell)$ restricts to $Q(\ell)$.
In this case, we also get P/Q.

Definition

A polynomial functor is a functor $\mathrm{Vec} \to \mathrm{Vec}$ obtained from T and the C_U via addition, multiplication, subfunctors and quotients.

Examples

- Square matrices: $V \mapsto V \otimes V$
- Tensors: $V \mapsto V \otimes \cdots \otimes V$
- Polynomials: $V \mapsto S^d V$

Polynomial functors: Topology

Definition

Let P, Q be polynomial functors. A morphism $\alpha\colon Q \to P$ is a family $(\alpha_V\colon Q(V) \to P(V))_{V\in\mathrm{Vec}}$ of polynomial maps such that

$$\begin{array}{ccc} Q(V) & \xrightarrow{\alpha_V} & P(V) \\ \downarrow{\scriptstyle Q(\ell)} & & \downarrow{\scriptstyle P(\ell)} \\ Q(W) & \xrightarrow{\alpha_W} & P(W) \end{array}$$

commutes for all linear maps $\ell\colon V \to W$.

Definition

A (closed) subset $X \subseteq P$ sends

$$V \mapsto \text{(closed) subset } X(V) \subseteq P(V)$$

such that $P(\ell)(X(V)) \subseteq X(W)$ for all linear maps $\ell\colon V \to W$.

Polynomial functors: Topology

Example

We have a morphism $C_{\mathbb{C}^{n\times(n-1)}} \oplus T^{n-1} \to T^n$ defined by:

$$\mathbb{C}^{n\times(n-1)} \oplus V^{n-1} \ni (A, v_1, \dots, v_{n-1}) \mapsto A \cdot (v_1, \dots, v_n)^\top \in V^n$$

Its image is the closed subset of T^n consisting of all linearly dependent n-tuples of vectors.

Example

We have a morphism $T^{2k} \to T \otimes T$ defined by:

$$V^{2k} \ni (v_1, w_1, \dots, v_k, w_k) \mapsto v_1 \otimes w_1 + \dots + v_k \otimes w_k \in V \otimes V$$

Its image is the closed subset of $T \otimes T$ consisting of all matrices of rank $\leq k$.

Polynomial functors: Topology

Example

We have a morphism $(S^1)^r \oplus (S^{d-1})^r \to S^d$ defined by:

$$(\ell_1, \ldots, \ell_r, g_1, \ldots, g_r) \mapsto \ell_1 \cdot g_1 + \ldots + \ell_r \cdot g_r$$

Its image is the closed subset of S^d consisting of all homogeneous polynomials of degree d and slice rank $\leq r$.

Example

The subset of $T^{\otimes n}$ consisting of tensors with tensor rank $\leq k$.

Example

The subset of S^d consisting of polynomials with strength $\leq r$.

Polynomial functors: The dichotomy

Let P, Q be polynomial functors. Write $Q \prec P$ when Q_d is a quotient of P_d for d maximal with $Q_d \ncong P_d$.

Dichotomy Theorem (B-Draisma-Eggermont-Snowden)

Let $X \subseteq P$ be a closed subset. Then $X = P$ or there are polynomial functors $Q_1, \ldots, Q_k \prec P$ and $\alpha_i \colon Q_i \to P$ such that $X \subseteq \bigcup_i \mathrm{im}(\alpha_i)$.

Consequence

Any closed subset of $T \otimes T$ consists of rank $\leq k \leq \infty$ matrices.

Consequence (B-Draisma-Eggermont)

Any closed subset of S^d consists of strength $\leq k$ polynomials.

Consequence (Draisma)

Any polynomial functor P is Noetherian.

Back to our goal

The homogeneous polynomials of degree 4 and strength ≤ 3 form a subset of S^4. This subset is the union of the images of the morphisms

$$\begin{array}{rcl} \alpha_k\colon (S^1 \oplus S^3)^{\oplus k} \oplus (S^2 \oplus S^2)^{\oplus 3-k} & \to & S^4 \\ ((\ell_i, q_i)_i, (g_j, h_j)_j) & \mapsto & \sum_{i=1}^{k} \ell_i \cdot q_i + \sum_{j=1}^{3-k} g_j \cdot h_j \end{array}$$

over $k = 0, 1, 2, 3$.

Goal

Prove that the subset $\bigcup_{k=0}^{3} \mathrm{im}(\alpha_k)$ of S^4 is not closed.

Idea

Consider polynomials of the form

$$x^2 f + y^2 g + u^2 h + v^2 q$$

with $x, y, u, v \in S^1$ and $f, g, h, q \in S^2$.

Back to our goal

Consider the morphism

$$\begin{array}{rcl} \beta\colon (S^1)^{\oplus 4} \oplus (S^2)^{\oplus 4} & \to & S^4 \\ (x,y,u,v,f,g,h,q) & \mapsto & x^2 f + y^2 g + u^2 h + v^2 q \end{array}$$

Lemma

We have $\mathrm{im}(\beta) \subseteq \overline{\mathrm{im}(\alpha_0)}$.

Proof.

The family of strength ≤ 3 polynomials

$$\frac{1}{t}\Big((x^2+tg)(y^2+tf)-(u^2-tq)(v^2-th)-(xy+uv)(xy-uv)\Big)$$

converges to $x^2 f + y^2 g + u^2 h + v^2 q$ as $t \to 0$. □

Goal

Prove that $\mathrm{im}(\beta) \not\subseteq \bigcup_{k=1}^{3} \mathrm{im}(\alpha_k)$.

Polynomial functors: Inverse limits

Let P be a polynomial functor.

Definition

We define P_∞ as the inverse limit of the sequence

$$\cdots \xrightarrow{P(\pi_4)} P(\mathbb{C}^4) \xrightarrow{P(\pi_3)} P(\mathbb{C}^3) \xrightarrow{P(\pi_2)} P(\mathbb{C}^2) \xrightarrow{P(\pi_1)} P(\mathbb{C}^1)$$

where $\pi_n \colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ is the projection forgetting the last coordinate.

Example

Take $P = T^n$. Then $P_\infty = (\mathbb{C}^{\mathbb{N}})^n$.

Example

Take $P = T \otimes T$. Then $P_\infty = \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$.

Polynomial functors: Inverse limits

A morphism $\alpha\colon Q \to P$ induces a map $\alpha_\infty\colon Q_\infty \to P_\infty$.

Example

The morphism $T^{2k} \to T \otimes T$ defined by

$$(v_1, w_1, \dots, v_k, w_k) \mapsto v_1 \otimes w_1 + \dots + v_k \otimes w_k$$

induces a map $(\mathbb{C}^{\mathbb{N}})^{2k} \to \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ (defined the same).

Let $p \in P_\infty$ be a point with projections $p_n \in P(\mathbb{C}^n)$.

Lemma

We have $p \in \mathrm{im}(\alpha_\infty)$ if and only if $p_n \in \mathrm{im}(\alpha_{\mathbb{C}^n})$ for all $n \geq 1$.

Proof.

Follows from a theorem by Lang stating that a countable system of polynomial equations over an uncountable field, any finite subsystem of which has a solution, has a solution. □

Polynomial functors: Systems of variables

Let P be a polynomial functor and $p \in P_\infty$ be a point.

Definition

We say that the point p is GL_∞-generic if $\overline{\mathrm{GL}_\infty \cdot p} = P_\infty$. Otherwise, the point is called degenerate.

Lemma

For $d \geq 2$, the set Ω_d of degenerate points in S^d_∞ equals the subspace of points with finite strength.

Proof.

Follows from the Dichotomy Theorem. □

Definition

A system of variables consists of a basis of S^d_∞/Ω_d over all $d \geq 1$.

Polynomial functors: Systems of variables

Let R, Q, P be direct sums of copies of S^d with $d \geq 1$.
Let $\beta\colon Q \to P$ and $\alpha\colon R \to P$ be morphisms.
Let $q \in Q_\infty$ and $r \in R_\infty$ be points.

Lemma

Suppose that q is GL_∞ generic and $p := \beta_\infty(q) = \alpha_\infty(r)$.
Then $\beta = \alpha \circ \gamma$ for some morphism $\gamma\colon Q \to R$.

Proof.

Extend q to a system of variables. Express r in these variables:

$$r = \delta(q, q'), \quad \delta\colon Q \oplus Q' \to R, \quad q' \in Q'_\infty$$

We have $\beta_\infty(q) = p = (\alpha \circ \delta)_\infty(q, q')$. So $p = (\alpha \circ \delta)_\infty(q, 0)$.

Take $\gamma = \delta(-, 0)$. Then $\beta = \alpha \circ \gamma$ since this holds on $\mathrm{GL}_\infty \cdot q$. □

The proof

We have the morphisms

$$\begin{aligned} \alpha_k\colon (S^1 \oplus S^3)^{\oplus k} \oplus (S^2 \oplus S^2)^{\oplus 3-k} &\to S^4 \\ ((\ell_i, q_i)_i, (g_j, h_j)_j) &\mapsto \sum_{i=1}^{k} \ell_i \cdot q_i + \sum_{j=1}^{3-k} g_j \cdot h_j \end{aligned}$$

for $k = 0, 1, 2, 3$ and the morphism

$$\begin{aligned} \beta\colon (S^1)^{\oplus 4} \oplus (S^2)^{\oplus 4} &\to S^4 \\ (x, y, u, v, f, g, h, q) &\mapsto x^2 f + y^2 g + u^2 h + v^2 q \end{aligned}$$

Goal

Prove that $\beta_\infty(x, y, u, v, f, g, h, q) \notin \bigcup_{k=0}^{3} \operatorname{im}(\alpha_{k,\infty})$.

Enough

Prove that $\beta = \alpha_k \circ \gamma$ has no solution for $k = 0, 1, 2, 3$.

The proof

Lemma

The equation $\beta = \alpha_0 \circ \gamma$ has no solution.

Proof.

We have to prove that

$$x^2 f + y^2 g + u^2 h + v^2 q \neq x_1 q_1 + x_2 q_2 + x_3 q_3$$

with x_i, q_i polynomials in x, y, u, v, f, g, h, q of degrees $1, 3$.

Coefficients of f, g, h, q on the left-hand side are x^2, y^2, u^2, v^2.

Coefficients of f, g, h, q on right-hand side are contained in the ideal $(x_1, x_2, x_3) \subseteq \mathbb{k}[x, y, u, v]$.

As $x^2, y^2, u^2, v^2 \in (x_1, x_2, x_3)$ cannot hold, we have inequality. □

The proof

Now we know that $\beta = \alpha_k \circ \gamma$ has no solution for $k = 0, 1, 2, 3$.

So $\beta_\infty(x, y, u, v, f, g, h, q) \notin \bigcup_{k=0}^{3} \mathrm{im}(\alpha_{k,\infty})$ for GL_∞-generic (x, y, u, v, f, g, h, q).

So $\beta_{\mathbb{C}^n}(x_n, y_n, u_n, v_n, f_n, g_n, h_n, q_n) \notin \bigcup_{k=0}^{3} \mathrm{im}(\alpha_{k,\mathbb{C}^n})$ for $n \gg 0$.

So the set

$$\{f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \mathrm{str}(f) \leq 3\}$$

is not Zariski-closed for $n \gg 0$.

Thanks for your attention!

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