Semi-algebraic properties of Minkowski sums of the bounded twisted cubic

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joint work with Adam Czapliński and Markus Wageringel

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Minkowski sums of the bounded twisted cubic

The Minkowski sum of n copies of the bounded twisted cubic is the set:

$$\mathcal{A}_{3,n} = \left\{ \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \end{pmatrix} + \cdots + \begin{pmatrix} t_{n-1} \\ t_{n-1}^2 \\ t_{n-1}^3 \end{pmatrix} + \begin{pmatrix} t_n \\ t_n^2 \\ t_n^3 \end{pmatrix} \middle| -1 \leqslant t_1, \ldots, t_n \leqslant 1 \right\} \subseteq \mathbb{R}^3$$

Problem (Rubinstein, Sarnak)

Determine $t_1, \ldots, t_n$ given a point in $\mathcal{A}_{3,n}$.

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Problem (Rubinstein, Sarnak)

Determine $t_1, \ldots, t_n$ *given a point in* $\mathcal{A}_{3,n}$.

Question (Rubinstein, Sarnak)

How can you efficiently determine whether a point $(x, y, z) \in \mathbb{R}^3$ *is an element of* $\mathcal{A}_{3,n}$*?*

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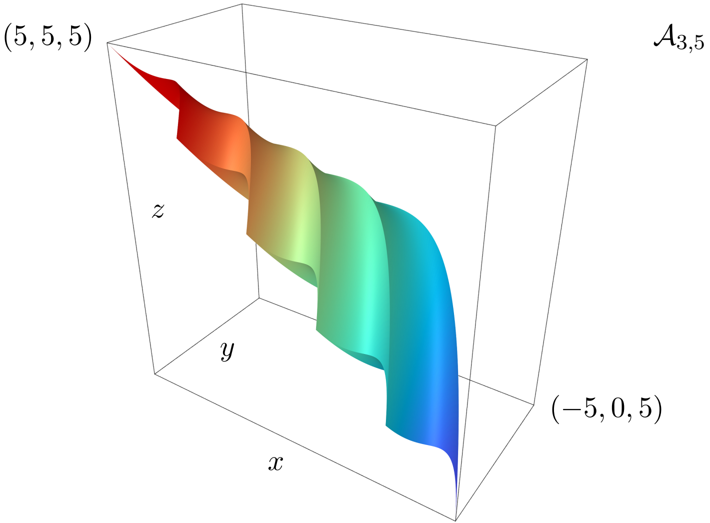
Question (Rubinstein, Sarnak)

How can you efficiently determine whether a point $(x, y, z) \in \mathbb{R}^3$ is an element of $\mathcal{A}_{3,n}$?

$\rightsquigarrow$ use a semi-algebraic description of $\mathcal{A}_{3,n}$ (polynomials, $=$, $\leqslant$, $\cup$)

Minkowski sums of the bounded twisted cubic



The semi-algebraic description

For $k, \ell > 0$, write

$$\begin{aligned}
A_{k\ell} &= k\ell(k+\ell)^2 > 0 \\
B_{k\ell}(x,y) &= 2k\ell x(2x^2 - 3(k+\ell)y) \\
C_{k\ell}(x,y) &= x^6 - 3(k+\ell)x^4 y + 3(k^2+k\ell+\ell^2)x^2y^2 - (k-\ell)^2(k+\ell)y^3 \\
f_{k\ell}(x,y,z) &= A_{k\ell}\cdot z^2 + B_{k\ell}(x,y)\cdot z + C_{k\ell}(x,y)
\end{aligned}$$

and take

$$X = \bigcup_{k=1}^{n-1} \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} + (n-k-1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} y \leqslant k + (x+k)^2 \\ y \geqslant (k+1)^{-1}x^2 \\ y \leqslant 1 + k^{-1}(x-1)^2 \\ z \leqslant \frac{-B_{k1}(x,y)}{2A_{k1}} \text{ or } f_{k1}(x,y,z) \leqslant 0 \end{array} \right\}$$

$$Y = \bigcup_{\ell=1}^{n-1} \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} + (n-\ell-1)\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} y \leqslant \ell + (x-\ell)^2 \\ y \geqslant (\ell+1)^{-1}x^2 \\ y \leqslant 1 + \ell^{-1}(x+1)^2 \\ z \geqslant \frac{-B_{1\ell}(x,y)}{2A_{1\ell}} \text{ or } f_{1\ell}(x,y,z) \leqslant 0 \end{array} \right\}$$

Then we have $\mathcal{A}_{3,n} = X \cap Y$.

Representations of points on the boundary

Let $p \in \mathbb{R}^3$ be a point on the boundary of $\mathcal{A}_{3,n}$ and write

$$p = \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \end{pmatrix} + \cdots + \begin{pmatrix} t_n \\ t_n^2 \\ t_n^3 \end{pmatrix}$$

for some tuple $(t_1, \ldots, t_n) \in [-1, 1]^n$.

Theorem

The set $\{t_1, \ldots, t_n\} \backslash \{-1, 1\}$ has at most two elements. And, the tuple $(t_1, \ldots, t_n)$ is unique up to permutation of its entries.

Representations of points on the boundary

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Proof.

Write

$$p = \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \end{pmatrix} + \cdots + \begin{pmatrix} t_n \\ t_n^2 \\ t_n^3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_1^2 \\ s_1^3 \end{pmatrix} + \cdots + \begin{pmatrix} s_n \\ s_n^2 \\ s_n^3 \end{pmatrix}$$

If $|t_i| < 1$, then $(1, 2t_i, 3t_i^2) \in T_p\,\mathcal{A}_{3,n}$.
If $|s_i| < 1$, then $(1, 2s_i, 3s_i^2) \in T_p\,\mathcal{A}_{3,n}$.

If $\#\{r, s, t\} = 3$, then

$$\begin{pmatrix} 1 & 1 & 1 \\ 2r & 2s & 2t \\ 3r^2 & 3s^2 & 3t^2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ r & s & t \\ r^2 & s^2 & t^2 \end{pmatrix}$$

has full rank.

Representations of points on the boundary

So the only possiblility is

$$\begin{pmatrix} n \\ p \end{pmatrix} = k \begin{pmatrix} 1 \\ s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} + a \begin{pmatrix} 1 \\ -1 \\ (-1)^2 \\ (-1)^3 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 1^2 \\ 1^3 \end{pmatrix}$$

for some fixed $-1 < s < t < 1$.

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So the only possiblility is

$$\binom{n}{p} = k \begin{pmatrix} 1 \\ s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} + a \begin{pmatrix} 1 \\ -1 \\ (-1)^2 \\ (-1)^3 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 1^2 \\ 1^3 \end{pmatrix}$$

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These vectors are linearly independent.
$\Rightarrow k, \ell, a, b$ are unique.

$\square$

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The set $\{t_1, \ldots, t_n\} \backslash \{-1, 1\}$ *has at most two elements. And, the tuple* $(t_1, \ldots, t_n)$ *is unique up to permutation of its entries.*

Semi-algebraic descriptions of boundary cells

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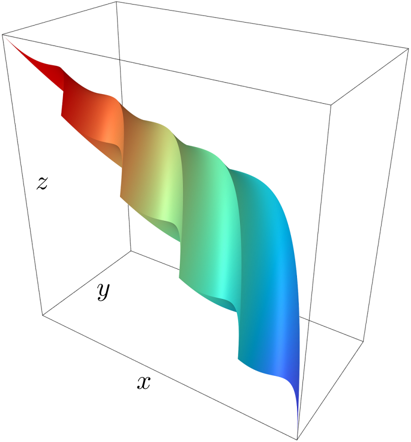
Theorem

Suppose that $k > \ell$. Then we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \;\middle|\; -1 \leqslant s \leqslant t \leqslant 1 \right\}$$

if and only if

$$\begin{aligned} 0 &\leqslant |x|, y, |z| \leqslant k + \ell \\ 0 &\leqslant k\ell D_{k\ell}(x, y) \leqslant k^2(k + \ell + x)^2, \ell^2(k + \ell - x)^2 \\ 0 &= A_{k\ell} \cdot z^2 + B_{k\ell}(x, y) \cdot z + C_{k\ell}(x, y) \\ z &\geqslant \frac{-B_{k\ell}(x, y)}{2A_{k\ell}} \end{aligned}$$



Thank you for your attention!

References

- Bik, Czapliński, Wageringel, *Semi-algebraic properties of Minkowski sums of the bounded twisted cubic*, in preparation.
- Rubinstein, Sarnak, *The underdetermined matrix moment Problem I*, in preparation.