ED Degrees of Orthogonally Invariant Varieties

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ED degree of a variety

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Fix a finite-dimensional complex vector space V,
a non-degenerate symmetric bilinear form on V,
a closed algebraic subvariety X of V (+ *conditions*).

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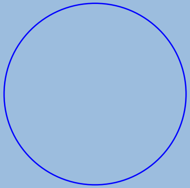
Then for a sufficiently general $v \in V$ the *positive* number

$$\#\left\{ x \in X^{\text{reg}} \,\middle|\, v - x \perp T_x X \right\}$$

is independent of v and is called the ED degree of X in V.

Example: unit circle

$$x^2 + y^2 = 1$$



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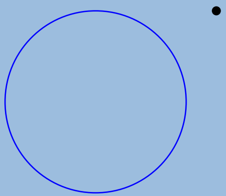

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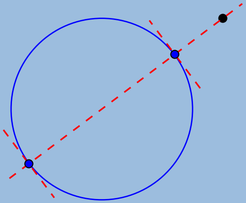

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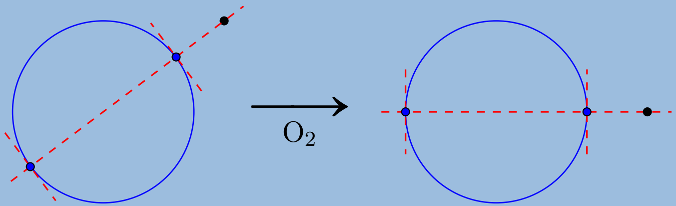

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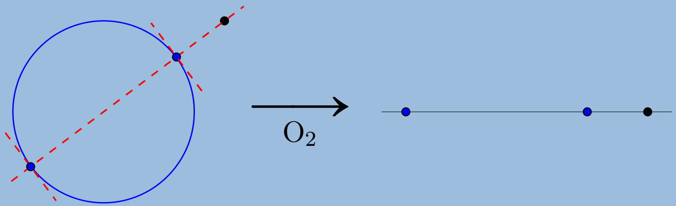

Orthogonally invariant matrix varieties

The group $\mathrm{O}(n) \times \mathrm{O}(m)$ acts on the space $\mathbb{C}^{n\times m}$ of $n \times m$ matrices. The bilinear form

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Let X be the closure in $\mathbb{C}^{n \times m}$ of a stable real subvariety of $\mathbb{R}^{n \times m}$ with smooth points and let X_0 be the subset of X of diagonal matrices. Then the ED degree of X in $\mathbb{C}^{n \times m}$ equals the ED degree of X_0 in the subspace of $\mathbb{C}^{n \times m}$ of all diagonal matrices.

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(2) For $D \in \mathbb{C}^{n\times m}$ a sufficiently general diagonal matrix, we have

$$\mathbb{C}^{n\times m} = \{\text{diagonal matrices}\} \oplus T_D\left(\mathrm{O}(n)D\mathrm{O}(m)\right).$$

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Let $V_0 \subseteq V$ be a subspace and set $X_0 := X \cap V_0$. Assume that GX_0 is dense in X and that

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Let $v \in V$ and $v_0 \in V_0$ be sufficiently general. We want:

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Lemma. Critical points of v_0 for X and X_0 are same.

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Example (Jiri Dadok)

Let $n > 0$ be an integer. Take $G = \mathrm{GL}_n$ acting on

$$V = \{(A, B) \in (\mathbb{C}^{n\times n})^2 | A = A^T, B = B^T\}$$

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$$V = V_0 \oplus T_{(D,D)} G(D, D)$$

for all invertible $D = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_i^2 \neq d_j^2$ for $i \neq j$.

Classification

Let G be reductive. Let K be a maximal compact subgroup of G and let $V_{\mathbb{R}}$ a real representation of K whose complexification is V.

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The following are equivalent:

(1) V *has a subspace* V_0 *such that*

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(2) V *is a stable polar representation.*

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Dadok classified irreducible polar representations of compact Lie groups.

Definition

A complex representation V of an reductive algebraic group G is stable polar if there is a vector $v \in V$, whose orbit is maximal-dimensional and closed, such that the subspace

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Definition

A real representation V of a compact Lie group K is polar if there is a vector $v \in V$, whose orbit is maximal-dimensional, such that for all $u \in (T_v Kv)^\perp$ we have $T_u Ku \subseteq T_v Kv$.

Classification

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Complexification of Dadok's list:

G	V
G semisimple	$\mathfrak{g}$
$\mathrm{O}(n)$	$\mathbb{C}^n$
$\mathrm{O}(n)$	$\mathrm{Sym}^2(\mathbb{C}^n)$
$\mathrm{O}(n) \times \mathrm{O}(m)$	$\mathbb{C}^{n\times m}$
$\mathrm{Sp}(n)$	$\Lambda^2(\mathbb{C}^{2n})$
$\mathrm{Sp}(n) \times \mathrm{Sp}(m)$	$\mathbb{C}^{2n\times 2m}$
$\mathrm{SL}(V)$	$V \oplus V^*$
$\mathrm{GL}(V)$	$\mathrm{Sym}^2(V) \oplus \mathrm{Sym}^2(V)^*$
$\mathrm{GL}(V)$	$\Lambda^2(V) \oplus \Lambda^2(V)^*$
$\mathrm{Sp}(n)$	$\mathbb{C}^{2n} \oplus (\mathbb{C}^{2n})^*$
$\mathrm{GL}_n \times \mathrm{GL}_m$	$\mathbb{C}^{n\times m} \oplus (\mathbb{C}^{n\times m})^*$
SL_2	$\mathrm{Sym}^4(\mathbb{C}^2)$
$\vdots$	$\vdots$

Thank you for your attention!

References

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