Curves with a rational MLE and chipfiring games

Arthur Bik

Max-Planck-Institut für Mathematik in den Naturwissenschaften

Seminar on Nonlinear Algebra

23 March 2022

Discrete statistical models

Let Δ_n be $\{(p_0, p_1, \ldots, p_n) \in \mathbb{R}^{n+1}_{>0} \mid p_0 + p_1 + \ldots + p_n = 1\}$.

Definition

A *discrete statistical model* is a subset $\mathcal{M}$ of Δ_n. The points of $\mathcal{M}$ represent probability distributions on the set $\{0, 1, \ldots, n\}$.

Definition

The *maximum likelihood estimator* (MLE) of $\mathcal{M}$ is the function

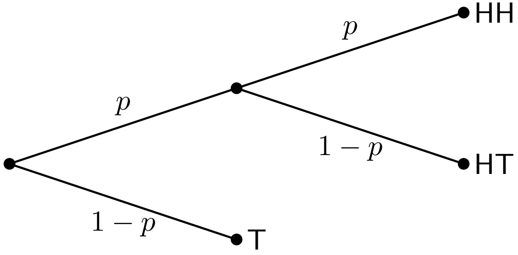
$$\Phi\colon \Delta_n \to \mathcal{M}$$

such that $(\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_n) = \Phi(u_0, u_n, \ldots, u_n)$ maximizes over $\mathcal{M}$ the chance that distribution $(u_0, u_1, \ldots, u_n)$ is observed from an experiment.

Discrete statistical models

Example

Flip a biased coin. When H flip again. Record the outcomes.



$$\mathcal{M} = \{(p^2, p(1-p), 1-p) \mid p \in (0,1)\}$$

Discrete statistical models

$$\mathcal{M} = \{(p^2, p(1-p), 1-p) \mid p \in (0,1)\}$$

Assume that $a+b+c$ experiments results in outcomes:

$$a \times \text{HH}, \quad b \times \text{HT}, \quad c \times \text{T}$$

What value of p maximizes the following?

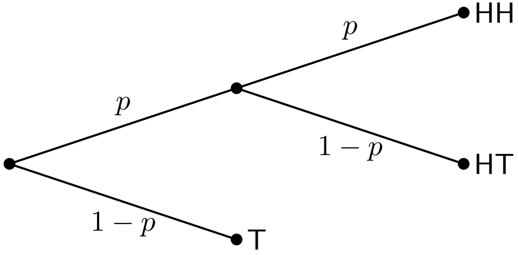
$$\binom{a+b+c}{a,b,c} \cdot (p^2)^a \cdot (p(1-p))^b \cdot (1-p)^c$$

$\rightsquigarrow$ $(2a+b)/\hat{p} - (b+2c)/(1-\hat{p}) = 0 \Rightarrow$

$$\hat{p} = \frac{2a+b}{2a+2b+c} \text{ and } 1-\hat{p} = \frac{b+c}{2a+2b+c}$$

Example

Flip a biased coin. When H flip again. Record the outcomes.



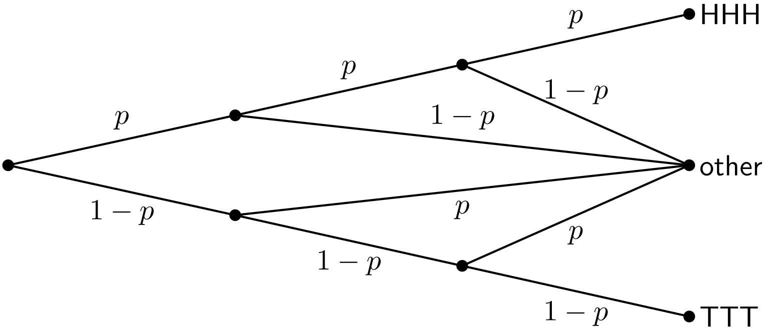
$$\mathcal{M} = \{(p^2, p(1-p), 1-p) \mid p \in (0,1)\}$$

$$\Phi(a,b,c) = \left(\left(\frac{2a+b}{2a+2b+c}\right)^2, \frac{2a+b}{2a+2b+c} \cdot \frac{b+c}{2a+2b+c}, \frac{b+c}{2a+2b+c}\right)$$

Discrete statistical models

Example

Flip a biased coin twice. When same outcomes flip again.
Record HHH, TTT or other.



$\mathcal{M} = \{(p^3, 3p(1-p), (1-p)^3) \mid p \in (0,1)\}$ and $\hat{p} = \dfrac{3a+b}{3a+2b+3c}$

Discrete statistical models with a rational MLE

Theorem (Duarte, Marigliano, Sturmfels)

The following are equivalent:

1. The model $\mathcal{M}$ has a rational MLE.
2. There exists a Horn pair (H, λ) such that $\mathcal{M}$ is the image of the Horn map.
3. There exists a discriminantal triple $(A, \Delta, \mathbf{m})$ such that $\mathcal{M}$ is the image of the associated map.

Question (Duarte, Marigliano, Sturmfels)

Can models with a rational MLE be classified?

Today (with Orlando Marigliano)

We focus on curves, i.e. models of dimension 1.

Curves with a rational MLE

Theorem

Let $\mathcal{M} \subseteq \Delta_n$ be a model of dimension 1 with a rational MLE. Then

$$\mathcal{M} = \{(\lambda_0 t^{i_0}(1-t)^{j_0}, \lambda_1 t^{i_1}(1-t)^{j_1}, \ldots, \lambda_n t^{i_n}(1-t)^{j_n}) \mid t \in (0,1)\}$$

for some $\lambda_\nu \in \mathbb{R}_{>0}$ and monomials $t^{i_\nu}(1-t)^{j_\nu}$ in $t, 1-t$ such that

$$\lambda_0 t^{i_0}(1-t)^{j_0} + \lambda_1 t^{i_1}(1-t)^{j_1} + \ldots + \lambda_n t^{i_n}(1-t)^{j_n} = 1$$

as polynomials.

Proof.

($\Leftarrow$) Compute the MLE.
($\Rightarrow$) Models with rational MLE are unirational. □

Curves with a rational MLE

Model consists of data $(\lambda_\nu, i_\nu, j_\nu)$ for $\nu = 0, \ldots, n$ such that

$$\lambda_0 t^{i_0}(1-t)^{j_0} + \lambda_1 t^{i_1}(1-t)^{j_1} + \ldots + \lambda_n t^{i_n}(1-t)^{j_n} = 1.$$

Reductions

1. If $(i_\nu, j_\nu) = (0,0)$, discard $(\lambda_\nu, i_\nu, j_\nu)$ and scale by $(1-\lambda_n u)^{-1}$.
2. If $(i_\nu, j_\nu) = (i_{\nu'}, j_{\nu'})$, combine them (by adding λ_ν and $\lambda_{\nu'}$).

We assume the model is *reduced*, i.e. all (i_ν, j_ν) distinct from $(0,0)$ and from each other.

Proposition

The data $(\lambda_\nu, i_\nu, j_\nu)$ for $\nu = 0, \ldots, n$ form a model $\Leftrightarrow$

$$-1 + \lambda_0 x^{i_0} y^{j_0} + \lambda_1 x^{i_1} y^{j_1} + \ldots + \lambda_n x^{i_n} y^{j_n} = (x+y-1) \sum_{i,j=0}^{\infty} f_{i,j} x^i y^j$$

for some $f_{i,j} \in \mathbb{R}$ almost all zero.

Chipsplitting games

Let $G = (V, E)$ be a (fixed) directed graph without loops.
Let $v_0 \in V$ have at least 1 outgoing edge $(v_0, v) \in E$.

Definition

1. A chip configuration is a tuple $w = (w_v)_{v \in V} \in \mathbb{Z}^V$.
2. A chipsplitting move at v_0 sends w to $\widetilde{w}$ defined by

$$\widetilde{w}_v = \begin{cases} w_v - 1 & \text{if } v = v_0, \\ w_v + 1 & \text{if } (v_0, v) \in E, \\ w_v & \text{otherwise} \end{cases}$$

   An unsplitting move at v_0 is its inverse.
3. The initial configuration w is given by $w_v = 0$ for all $v \in V$.
4. A chipsplitting game f is a finite sequence of moves.
5. The outcome of f is the result of applying all moves starting from the initial configuration.

Chipsplitting games

Let $d \in \{1, 2, 3, \dots, \infty\}$. Define

$$\begin{aligned} V_d &:= \{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid \deg(i,j) \leq d\} \\ E_d &:= \{(v, v+e) \mid v \in V_{d-1}, e \in \{(1,0),(0,1)\}\} \end{aligned}$$

where $\deg(i,j) := i + j$.

Example

We apply a splitting move at the red vertex.

$$\begin{array}{cccccc} 0 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Chipsplitting games

Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

$$\begin{array}{rcl} V_d & := & \{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid \deg(i,j) \leq d\} \\ E_d & := & \{(v, v+e) \mid v \in V_{d-1}, e \in \{(1,0),(0,1)\}\} \end{array}$$

where $\deg(i,j) := i + j$.

Example

We apply a splitting move at the red vertex.

$$\begin{array}{cccccc} 0 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & \\ 1 & 0 & 0 & 0 & 0 & \\ -1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

$$\begin{array}{rcl} V_d & := & \{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid \deg(i,j) \leq d\} \\ E_d & := & \{(v, v+e) \mid v \in V_{d-1}, e \in \{(1,0),(0,1)\}\} \end{array}$$

where $\deg(i,j) := i + j$.

Example

We apply a splitting move at the red vertex.

$$\begin{array}{rrrrrr} 0 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & 0 & \\ -1 & {\color{red}1} & 0 & 0 & 0 & 0 \end{array}$$

Chipsplitting games

Let $d \in \{1, 2, 3, \dots, \infty\}$. Define

$$\begin{array}{rcl} V_d & := & \{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid \deg(i,j) \leq d\} \\ E_d & := & \{(v, v+e) \mid v \in V_{d-1}, e \in \{(1,0),(0,1)\}\} \end{array}$$

where $\deg(i,j) := i + j$.

Example

We apply a splitting move at the red vertex.

$$\begin{array}{cccccc} 0 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 2 & 0 & 0 & 0 & \\ -1 & 0 & 1 & 0 & 0 & 0 \end{array}$$

Chipsplitting games

Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

$$\begin{aligned} V_d &:= \{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid \deg(i,j) \leq d\} \\ E_d &:= \{(v, v+e) \mid v \in V_{d-1}, e \in \{(1,0),(0,1)\}\} \end{aligned}$$

where $\deg(i,j) := i + j$.

Example

We apply a splitting move at the red vertex.

0					
0	0				
1	0	0			
0	1	0	0		
0	2	0	0	0	
−1	0	1	0	0	0

Chipsplitting games

Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

$$\begin{aligned} V_d &:= \{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid \deg(i,j) \leq d\} \\ E_d &:= \{(v, v+e) \mid v \in V_{d-1}, e \in \{(1,0),(0,1)\}\} \end{aligned}$$

where $\deg(i,j) := i + j$.

Example

We apply a splitting move at the red vertex.

$$\begin{matrix} 0 \\ 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \color{red}{2} & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

Chipsplitting games

Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

$$\begin{aligned} V_d &:= \{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid \deg(i,j) \leq d\} \\ E_d &:= \{(v, v+e) \mid v \in V_{d-1}, e \in \{(1,0),(0,1)\}\} \end{aligned}$$

where $\deg(i,j) := i + j$.

Example

We apply a splitting move at the red vertex.

$$\begin{matrix} 0 \\ 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

Curves with a rational MLE as outcomes

Proposition

The data $(\lambda_\nu, i_\nu, j_\nu)$ for $\nu = 0, \ldots, n$ form a model $\Leftrightarrow$

$$-1+\lambda_0 x^{i_0} y^{j_0} + \lambda_1 x^{i_1} y^{j_1} + \ldots + \lambda_n x^{i_n} y^{j_n} = (x+y-1) \sum_{i,j=0}^{\infty} f_{i,j} x^i y^j$$

for some $f_{i,j} \in \mathbb{R}$ almost all finite.

Assume the model is reduced and set

$$w_{i,j} = \begin{cases} \lambda_\nu & \text{if } (i,j) = (i_\nu, j_\nu), \\ -1 & \text{if } (i,j) = (0,0), \\ 0 & \text{otherwise} \end{cases}$$

Then $(w_{i,j})_{(i,j)\in V_d}$ is the outcome of the chipsplitting game where (i,j) is split $f_{i,j}$ times (where unsplitting moves count negatively).

Chipsplitting games

Definition

1. A chip configuration w is *valid* when $w_{i,j} \geq 0$ for all $(i,j) \neq (0,0)$.
2. The *positive support* of w is $\operatorname{supp}^+(w) := \{(i,j) \mid w_{i,j} > 0\}$.
3. The *degree* of w is $\deg(w) := \max\{\deg(i,j) \mid w_{i,j} \neq 0\}$.

Conjecture

Let w be a valid outcome. Then $\deg(w) \leq 2 \cdot \# \operatorname{supp}^+(w) - 3$.

Why is the nice?

- The conjecture gives a bound of the degree of the parametrisation of a dimension-1 curve with a rational MLE.
- The conjecture shows that there are finitely many "fundamental" models in Δ_n, which can be used to get any other model.

Composite models

Definition

A model $\{(\lambda_\nu, i_\nu, j_\nu) \mid \nu = 0, \dots, n\}$ is fundamental when the λ_ν are unique given the i_ν, j_ν.

Composition

Let $\mu \in (0,1)$. The μ-composite of models

$$\{(\lambda_{i,j}, i, j) \mid (i,j) \in S\}, \quad \{(\lambda'_{i,j}, i, j) \mid (i,j) \in S'\}$$

is the model

$$\{(\lambda_{i,j} + \lambda'_{i,j}, i, j) \mid (i,j) \in S \cup S'\}$$

where $\lambda_{i,j} := 0$ for all $(i,j) \notin S$ and $\lambda'_{i,j} := 0$ for all $(i,j) \notin S'$.

Theorem

Every reduced model in Δ_n is a composite of $\leq n$ fundamental models (from Δ_m with $m < n$).

Chipsplitting games

Conjecture

Let w be a valid outcome. Then $\deg(w) \leq 2 \cdot \#\operatorname{supp}^+(w) - 3$.

Why believe the conjecture?

- Computer search for low degree. $(\frac{1}{2}(\deg(w) + 3) \leq \#\operatorname{supp}^+(w))$
- Take $d = 2k + 1$. Let $w = (w_{i,j})_{(i,j)\in V_d} \in \mathbb{Z}^{V_d}$ be defined by

$$\begin{aligned} w_{0,0} &= -1, \\ w_{0,2k+1} &= 1, \\ w_{2i+1,k-i} &= \frac{2k+1}{2i+1}\binom{k+i}{2i}, \quad i \in \{0, 1, \ldots, k\} \end{aligned}$$

and $w_{i,j} = 0$ otherwise. Then w is a valid outcome.

$\deg(w) = 2k + 1 = 2 \cdot (k + 2) - 3 = 2 \cdot \#\operatorname{supp}^+(w) - 3$

Main results

Conjecture

Let w be a valid outcome. Then $\deg(w) \leq 2 \cdot \#\,\mathrm{supp}^+(w) - 3$.

Main result

The conjecture holds when $\#\,\mathrm{supp}^+(w) \leq 5$.

Corollary

Let

$$\mathcal{M} = \{(\lambda_0 t^{i_0}(1-t)^{j_0}, \lambda_1 t^{i_1}(1-t)^{j_1}, \ldots, \lambda_n t^{i_n}(1-t)^{j_n}) \mid t \in (0,1)\}$$

be a model with a rational MLE.

1. If $n = 1$, then $\max_\nu (i_\nu + j_\nu) \leq 1$.
2. If $n = 2$, then $\max_\nu (i_\nu + j_\nu) \leq 3$.
3. If $n = 3$, then $\max_\nu (i_\nu + j_\nu) \leq 5$.
4. If $n = 4$, then $\max_\nu (i_\nu + j_\nu) \leq 7$.

The proof

Conjecture

Let w be a valid outcome. Then $\deg(w) \leq 2 \cdot \#\operatorname{supp}^+(w) - 3$.

We aim to prove that certain chip configurations cannot be the outcome of a chipsplitting game.

Here are the tools:

1. Invertibility Criterium
2. Hyperfield Criterium
3. Hexagon Criterium
4. a computer

Pascal equations

For $(k,\ell) \in V_{d-1}$, take $E^{(k,\ell)} \in \mathbb{Z}_d^V$ so that

$$E_{i,j}^{(k,\ell)} = \begin{cases} 1 & \text{when } (i,j) \in \{(k+1,\ell),(k,\ell+1)\}, \\ -1 & \text{when } (i,j) = (k,\ell), \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathrm{span}_{\mathbb{Z}}\{E^{(k,\ell)} \mid (k,\ell) \in V_{d-1}\}$ is the space of outcomes.

Definition

A *Pascal equation* on $\mathbb{Z}^{V_d}$ is a linear form

$$\sum_{(i,j)\in V_d} c_{i,j} x_{i,j}$$

such that $c_{i,j} = c_{i+1,j} + c_{i,j+1}$ for all $(i,j) \in V_{d-1}$.

We have $\{\text{outcomes}\} = V(\text{Pascal equations})$.

The Invertibility Criterium

For $a, b \geq 0$ with $a + b = d$, define

$$\varphi_{a,b} := \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{d-(i+j)}{a-i} x_{i,j} = \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{d-(i+j)}{b-j} x_{i,j}$$

For $w \in \mathbb{Z}^{V_d}$, define $\operatorname{supp}(w) := \{(i,j) \in V_d \mid w_{i,j} \neq 0\} \subseteq V_d$.

Invertibility Criterium

Let $S \subseteq V_d$ and $E \subseteq \{(a,b) \in V_d \mid a + b = d\}$ be subsets of the same size. Let $w \in \mathbb{Z}^{V_d}$ be an outcome. Suppose that the matrix

$$A_{E,S} := \left(\binom{d-(i+j)}{a-i} \right)_{a \in E, (i,j) \in S}$$

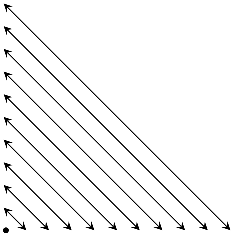
is invertible. Then $\operatorname{supp}(w) \neq S$.

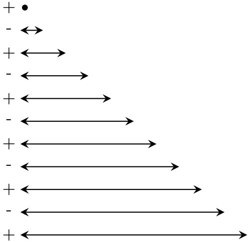
The Invertibility Criterium

How to apply it?

1. Split into pieces.
2. Use symmetry:
   We have an action of S_3 on $\mathbb{Z}^{V_d}$ given by

$$
\begin{aligned}
(12) \cdot (w_{i,j})_{(i,j)\in V_d} &:= (w_{j,i})_{(i,j)\in V_d} \\
(13) \cdot (w_{i,j})_{(i,j)\in V_d} &= ((-1)^{d-j} w_{d-(i+j),j})_{(i,j)\in V_d}
\end{aligned}
$$





The Hyperfield Criterium

Definition

A *hyperfield* is a tuple $(H, +, \cdot, 0, 1)$ where ...

Example (Sign hyperfield)

Take $H = \{1, 0, -1\}$ with usual multiplication and

$$s + r := \{\mathrm{sign}(x + y) \mid x, y \in \mathbb{R}, \mathrm{sign}(x) = s, \mathrm{sign}(y) = r\}$$

for all $s, r \in H$.

We have $0 + s = s$, $s + s = s$ and $1 + (-1) = H$.

The Hyperfield Criterium

Definition

A *hyperfield* is a tuple $(H, +, \cdot, 0, 1)$ where

$$- + -\colon H \times H \to 2^H \setminus \{\emptyset\}, \quad - \cdot -\colon H \times H \to H$$

are symmetric maps satisfying the following relations:

1. The tuple $(H \setminus \{0\}, \cdot, 1)$ is a group.
2. We have $0 \cdot x = 0$ and $0 + x = \{x\}$ for all $x \in H$.
3. We have $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$ for all $a, x, y \in H$.
4. For every $x \in H$ there is an unique element $-x \in H$ such that $x + (-x) \ni 0$.

A subset of H^n is Zariski-closed when it is of the form

$$\{(s_1, \dots, s_n) \in H^n \mid f_1(s_1, \dots, s_n), \dots, f_k(s_1, \dots, s_n) \ni 0\}$$

for some polynomials $f_1, \dots, f_k$ over H in variables $x_1, \dots, x_n$.

Example (Sign hyperfield)

Take $H = \{1, 0, -1\}$ with usual multiplication and

$$0 + s = s, \quad s + s = s, \quad 1 + (-1) = H$$

Take $f = x_1 + x_2 - x_3 - x_4$ and $s_1, s_2, s_3, s_4 \in H$. Then

$$f(s_1, s_2, s_3, s_4) \ni 0 \quad \Leftrightarrow \quad \begin{cases} s_1 = s_2 = s_3 = s_4 = 0 \\ \text{or} \\ 1, -1 \in \{s_1, s_2, -s_3, -s_4\} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} f(s_1, s_2, s_3, s_4) = 0 \\ \text{or} \\ f(s_1, s_2, s_3, s_4) = H \end{cases}$$

The Hyperfield Criterium

For $f = \sum_i c_i x_i \in \mathbb{R}[x_1, \ldots, x_n]$, take $\mathrm{sign}(f) := \sum_i \mathrm{sign}(c_i) x_i$.

Hyperfield Criterium

Let $w \in \mathbb{Z}^{V_d}$ be an outcome and $s \in H^{V_d}$. Suppose that $\mathrm{sign}(\phi)$ does not vanish at s for some Pascal equation ϕ on $\mathbb{Z}^{V_d}$. Then $\mathrm{sign}(w) \neq s$.

How to apply it?

The Hexagon Criterium

Let $\ell_1, \ell_2 \geq d' \geq 1$ be integers such that $d' + \ell_1 + \ell_2 \leq d$.

Let $w = (w_{i,j})_{(i,j)\in V_d} \in \mathbb{Z}^{V_d}$ and write $w' = (w_{i,j})_{(i,j)\in V_{d'}} \in \mathbb{Z}^{V_{d'}}$.

Hexagon Criterium

Suppose that w' is not an outcome and

$$\operatorname{supp}(w) \subseteq V_{d'} \cup \{(i,j) \in V_d \mid j > d-\ell_1\} \cup \{(i,j) \in V_d \mid i > d-\ell_2\}$$

holds. Then w is not an outcome.

How to apply it?

Main results

Conjecture

Let w be a valid outcome. Then $\deg(w) \leq 2 \cdot \#\operatorname{supp}^+(w) - 3$.

Main result

The conjecture holds when $\#\operatorname{supp}^+(w) \leq 5$.

Corollary

Let

$$\mathcal{M} = \{(\lambda_0 t^{i_0}(1-t)^{j_0}, \lambda_1 t^{i_1}(1-t)^{j_1}, \ldots, \lambda_n t^{i_n}(1-t)^{j_n}) \mid t \in (0,1)\}$$

be a model with a rational MLE.

1. If $n = 1$, then $\max_\nu(i_\nu + j_\nu) \leq 1$.
2. If $n = 2$, then $\max_\nu(i_\nu + j_\nu) \leq 3$.
3. If $n = 3$, then $\max_\nu(i_\nu + j_\nu) \leq 5$.
4. If $n = 4$, then $\max_\nu(i_\nu + j_\nu) \leq 7$.

Main results

Conjecture

Let w be a valid outcome. Then $\deg(w) \leq 2 \cdot \#\operatorname{supp}^+(w) - 3$.

Main result

The conjecture holds when $\#\operatorname{supp}^+(w) \leq 5$.

Corollary

Let

$$\mathcal{M} = \{(\lambda_0 t^{i_0}(1-t)^{j_0}, \lambda_1 t^{i_1}(1-t)^{j_1}, \ldots, \lambda_n t^{i_n}(1-t)^{j_n}) \mid t \in (0,1)\}$$

be a model with a rational MLE.

1. If $n = 1$, then $\max_\nu(i_\nu + j_\nu) \leq 1$. $\Leftarrow$ Invertibility Criterium
2. If $n = 2$, then $\max_\nu(i_\nu + j_\nu) \leq 3$. $\Leftarrow$ Invertibility Criterium
3. If $n = 3$, then $\max_\nu(i_\nu + j_\nu) \leq 5$.
4. If $n = 4$, then $\max_\nu(i_\nu + j_\nu) \leq 7$.

Main results

Conjecture

Let w be a valid outcome. Then $\deg(w) \leq 2 \cdot \#\operatorname{supp}^+(w) - 3$.

Main result

The conjecture holds when $\#\operatorname{supp}^+(w) \leq 5$.

Corollary

Let

$$\mathcal{M} = \{(\lambda_0 t^{i_0}(1-t)^{j_0}, \lambda_1 t^{i_1}(1-t)^{j_1}, \ldots, \lambda_n t^{i_n}(1-t)^{j_n}) \mid t \in (0,1)\}$$

be a model with a rational MLE.

1. If $n = 1$, then $\max_\nu(i_\nu + j_\nu) \leq 1$. $\Leftarrow$ Invertibility Criterium
2. If $n = 2$, then $\max_\nu(i_\nu + j_\nu) \leq 3$. $\Leftarrow$ Invertibility Criterium
3. If $n = 3$, then $\max_\nu(i_\nu + j_\nu) \leq 5$. $\Leftarrow$ Hyperfield Criterium
4. If $n = 4$, then $\max_\nu(i_\nu + j_\nu) \leq 7$.

Main results

Conjecture

Let w be a valid outcome. Then $\deg(w) \leq 2 \cdot \#\operatorname{supp}^+(w) - 3$.

Main result

The conjecture holds when $\#\operatorname{supp}^+(w) \leq 5$.

Corollary

Let

$$\mathcal{M} = \{(\lambda_0 t^{i_0}(1-t)^{j_0}, \lambda_1 t^{i_1}(1-t)^{j_1}, \ldots, \lambda_n t^{i_n}(1-t)^{j_n}) \mid t \in (0,1)\}$$

be a model with a rational MLE.

1. If $n = 1$, then $\max_\nu(i_\nu + j_\nu) \leq 1$. $\Leftarrow$ Invertibility Criterium
2. If $n = 2$, then $\max_\nu(i_\nu + j_\nu) \leq 3$. $\Leftarrow$ Invertibility Criterium
3. If $n = 3$, then $\max_\nu(i_\nu + j_\nu) \leq 5$. $\Leftarrow$ Hyperfield Criterium
4. If $n = 4$, then $\max_\nu(i_\nu + j_\nu) \leq 7$. $\Leftarrow$ HypC + InvC + HexC

Main results

Conjecture

Let w be a valid outcome. Then $\deg(w) \leq 2 \cdot \#\operatorname{supp}^+(w) - 3$.

Some computations

The conjecture holds when $\deg(w) \leq 9$.

$n \backslash d$	1	2	3	4	5	6	7	8	9
2	1	–	–	–	–	–	–	–	–
3	–	3	1	–	–	–	–	–	–
4	–	–	12	4	2	–	–	–	–
5	–	–	–	82	38	10	4	–	–
6	–	–	–	–	602	254	88	24	2

$\#\{$"fundamental" outcomes with $\#\operatorname{supp}^+(w) = n, \deg(w)) = d\}$

Thank you for your attention!

References

Eliana Duarte, Orlando Marigliano, Bernd Sturmfels
Discrete statistical models with rational maximum likelihood estimator
Bernoulli 27 (2021), pp. 135–154

Arthur Bik, Orlando Marigliano
Discrete statistical curves with rational maximum likelihood estimator
in preparation