Strength of Polynomials

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Seminar on Nonlinear Algebra

13 August 2021

MAX-PLANCK-GESELLSCHAFT

The strength of polynomials

Let f be a homogeneous polynomial of degree $d \geq 2$ over $\mathbb{C}$.

Definition

The *strength* of f is the minimal number $\mathrm{str}(f) := r \geq 0$ such that

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

with $g_1, h_1, \ldots, g_r, h_r$ homogeneous polynomials of degree $\leq d - 1$.

First examples

(0) $\mathrm{str}(f) = 0 \Leftrightarrow f = 0$

(1) $\mathrm{str}(f) = 1 \Leftrightarrow f \neq 0$ is reducible

(2) $\mathrm{str}(f) \geq 2 \Leftrightarrow f$ is irreducible

Example

The polynomial

$$x^2 + y^2 + z^2 = (x + iy) \cdot (x - iy) + z \cdot z$$

has strength 2. (It would be 3 over $\mathbb{R}$.)

Why care about strength?

Reason 1 - Data efficiency

A homogeneous polynomial of degree d in $n+1$ variables has

$$\binom{n+d}{d}$$

coefficients.

A polynomial f of degree 3 in 10^6 variables has

$$\approx 1.67 \cdot 10^{17}$$

coefficients.

The number of coefficients in a strength decomposition is

$$\approx \operatorname{str}(f) \cdot 5.00 \cdot 10^{11}.$$

So saving this uses $\approx 33400/\operatorname{str}(f)$ times less space.

Why care about strength?

Reason 2 - Universality

Let $f \in \mathbb{C}[x_1, \dots, x_n]_d$ be a homogeneous polynomial.

For $c_{11}, \dots, c_{nm} \in \mathbb{C}$, the polynomial
$f(c_{11}y_1 + \ldots + c_{1m}y_m, \dots, c_{n1}y_1 + \ldots + c_{nm}y_m) \in \mathbb{C}[y_1, \dots, y_m]_d$
is a coordinate transformation of f.

Theorem (Kazhdan-Ziegler, B-Draisma-Eggermont)

Let $\mathcal{P}$ be a property of degree-d polynomials such that

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{every coordinate transformation of } f \text{ has } \mathcal{P}$$

and not every polynomial has $\mathcal{P}$. Then the exists a $k \geq 0$ such that

$$f \text{ has } \mathcal{P} \Rightarrow \mathrm{str}(f) \leq k$$

Why care about strength?

Reason 2 - Universality

Let $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ be a homogeneous polynomial.

For $c_{11}, \ldots, c_{nm} \in \mathbb{C}$, the polynomial
$f(c_{11}y_1 + \ldots + c_{1m}y_m, \ldots, c_{n1}y_1 + \ldots + c_{nm}y_m) \in \mathbb{C}[y_1, \ldots, y_m]_d$
is a coordinate transformation of f.

Example (Kazhdan-Ziegler)

For every $\ell \geq 0$, there exists a $k \geq 0$ such that

all partial derivatives of f have strength $\leq \ell \Rightarrow \mathrm{str}(f) \leq k$

This property satisfies the condition because of the chain rule.

Why care about strength?

Reason 3 - It is like the rank of matrices

We have a one-to-one correspondence

$$\begin{array}{rcl} \{A \in \mathbb{C}^{n\times n} \mid A = A^\top\} & \leftrightarrow & \mathbb{C}[x_1,\dots,x_n]_2 \\ A & \mapsto & (x_1,\dots,x_n)A(x_1,\dots,x_n)^\top \\ (a_1,\dots,a_n)^\top(a_1,\dots,a_n) & \mapsto & (a_1x_1+\ldots+a_nx_n)^2 \\ vw^\top + wv^\top & \mapsto & 2\cdot(x_1,\dots,x_n)v\cdot(x_1,\dots,x_n)w \end{array}$$

Write $f = (x_1,\dots,x_n)A(x_1,\dots,x_n)^\top$. Then

$$\begin{array}{rcl} \mathrm{str}(f) \le k & \Leftrightarrow & f \text{ is a sum of } k \text{ reducible polynomials} \\ & \Leftrightarrow & A \text{ is a sum of } k \text{ matrices of rank} \le 2 \\ & \Leftrightarrow & A \text{ has rank} \le 2k \end{array}$$

So $\mathrm{str}(f) = \lceil \mathrm{rk}(A)/2 \rceil$.

Example

$\mathrm{str}(x^2+y^2+z^2) = \lceil \mathrm{rk}(I_3)/2 \rceil = 2.$

Basic properties of strength

How does strength compare to rank of matrices?

We can compute the rank of a matrix.
(determinants of submatrices / column- and rowoperations)
Q: How do you compute the strength of a polynomial?

The limit of a sequence of matrices of rank $\leq k$ has rank $\leq k$.
Q: Is the subset of polynomials of strength $\leq k$ closed?

An $n \times m$ matrix has maximal rank $\min(n, m)$.
Q: What is the maximal strength of a polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$?

A random $n \times m$ matrix has rank $\min(n, m)$.
Q: What is the strength of a random polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$?

Computing the strength of a polynomial

I don't know how to do this... **Exercise** Find an algorithm.

Tricks

1. We have $\operatorname{str}(f+g) \leq \operatorname{str}(f) + \operatorname{str}(g)$.
2. For $f \in \mathbb{C}[x_1, \ldots, x_n]_d$, we define the singular locus:
$$\operatorname{Sing}(f) := \left\{ \frac{\partial f}{\partial x_1} = \ldots = \frac{\partial f}{\partial x_n} = 0 \right\}$$
When $f = g_1 \cdot h_1 + \ldots + g_k \cdot h_k$, then
$$\{g_1 = h_1 = \ldots = g_k = h_k = 0\} \subseteq \operatorname{Sing}(f)$$
and so $\operatorname{codim} \operatorname{Sing}(f) := n - \dim \operatorname{Sing}(f) \leq 2k$.
3. Every polynomial in $\mathbb{C}[x,y]_d$ is reducible. Hence
$$f \in \mathbb{C}[x,y]_d \Rightarrow \operatorname{str}(f) \leq 1$$

Computing the strength of a polynomial

Example

Consider $f = x_1^d + \ldots + x_n^d$.

We have

$$f = \begin{cases} (x_1^d + x_2^d) + \ldots + (x_{2k-1}^2 + x_{2k}^d) & \text{if } n = 2k \\ (x_1^d + x_2^d) + \ldots + (x_{2k-1}^2 + x_{2k}^d) + x_{2k+1}^d & \text{if } n = 2k+1 \end{cases}$$

and so $\mathrm{str}(f) \leq \lceil n/2 \rceil$.

The singular locus

$$\mathrm{Sing}(f) = \{dx_1^{d-1} = \ldots = dx_n^{d-1} = 0\} = \{(0, \ldots, 0)\} \subseteq \mathbb{C}^n$$

has codimenion n. So $\mathrm{str}(f) \geq \lceil n/2 \rceil$.

So $\mathrm{str}(f) = \lceil n/2 \rceil$.

Strength ≤ 3 is not closed

Q: Is the subset of polynomials of strength $\leq k$ closed?

- For $k = 1$, this is the set of reducible polynomials.

$$\begin{aligned} \mathbb{P}(\mathbb{C}[x_1, \ldots, x_n]_i) \times \mathbb{P}(\mathbb{C}[x_1, \ldots, x_n]_{d-i}) &\rightarrow \mathbb{P}(\mathbb{C}[x_1, \ldots, x_n]_d) \\ ([g], [h]) &\mapsto [g \cdot h] \end{aligned}$$

  has closed image.
- For $k = 2$, I don't know. (Conjecture: yes)
- For $d = 2$, this is the set of symmetric matrices of rank $\leq 2k$.
- For $d = 3$, this is true. (slice rank of polynomials)

Theorem (Ballico-B-Oneto-Ventura)

The set of polynomials in $\mathbb{C}[x_1, \ldots, x_n]_4$ of strength ≤ 3 is not closed for $n \gg 0$.

Strength ≤ 3 is not closed

For $t \neq 0$ and f, g, p, q of degree 2 and x, y, u, v variables, the polynomial

$$ {}^1/_t \Big((x^2 + tg)(y^2 + tf) - (u^2 - tq)(v^2 - tp) - (xy - uv)(xy + uv) \Big) $$

$$ = $$

$$ x^2 f + y^2 g + u^2 p + v^2 q + t(fg - pq) $$

has strength ≤ 3. For $t \to 0$, we get

$$ x^2 f + y^2 g + u^2 p + v^2 q $$

Theorem (Ballico-B-Oneto-Ventura)

For $n \gg 0$, there are polynomials $f, g, p, q \in \mathbb{C}[z_1, \ldots, z_n]_2$ such that

$$ x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, z_1, \ldots, z_n]_4 $$

has strength 4.

And now for something completely different

Consider the polynomial

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

where x, y, u, v have degree 1 and $\underbrace{f, g, p, q}_{\text{variables}}$ have degree 2.

Definition

For $d \geq 2$, the strength of a polynomial $h \in \mathbb{C}[x, y, u, v, f, g, p, q]_d$ is the minimum number $r \geq 0$ (when this exists) such that

$$h = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

with $g_1, h_1, \ldots, g_r, h_r$ homogeneous polynomials of degree $\leq d - 1$.

Example

When the g_i, h_i are homogeneous polynomials of degree ≤ 1, then

$$g_1 \cdot h_1 + \ldots + g_r \cdot h_r \in \mathbb{C}[x, y, u, v]$$

Hence the variable f has infinite strength.

And now for something completely different

Proposition

The polynomial

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

has strength 4.

1/4 of the proof

We need to show, for example, that

$$x^2 f + y^2 g + u^2 p + v^2 q \neq \ell_1 \cdot h_1 + \ell_2 \cdot h_2 + \ell_3 \cdot h_3$$

for all $\ell_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_1$ and $h_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_3$.

And now for something completely different

Proposition

The polynomial

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

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for all $\ell_i \in \mathbb{C}[x, y, u, v]_1$ and $h_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_3$.

Think of $R = \mathbb{C}[x, y, u, v]$ as the set of coefficients.
So $\ell_i \in R$ and $h_i \in R[f, g, p, q]$.

The coefficients of f, g, p, q on the right are all in (ℓ_1, ℓ_2, ℓ_3).
The coefficients x^2, y^2, u^2, v^2 on the left are not all (ℓ_1, ℓ_2, ℓ_3). $\square$

Strength ≤ 3 is not closed

Theorem (Ballico-B-Oneto-Ventura)

For $n \gg 0$, there are polynomials $f, g, p, q \in \mathbb{C}[z_1, \dots, z_n]_2$ such that

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, z_1, \dots, z_n]_4$$

has strength 4.

Proposition

The polynomial

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

has strength 4.

Fact. The proposition implies the theorem.

The proof uses the geometry of polynomial functors.

Generic and maximal strength

Q: What is the maximal strength of a polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$?

Q: What is the strength of a random polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$?

Definition

The *slice rank* of f is the minimal $\operatorname{slrk}(f) := r \geq 0$ such that

$$f = \ell_1 \cdot h_1 + \ldots + \ell_r \cdot h_r$$

with $\ell_1, \ldots, \ell_r$ and $h_1, \ldots, h_r$ homogeneous of degrees 1 and $d-1$.

Proposition

1. $\operatorname{str}(f) \leq \operatorname{slrk}(f) \leq n-1$
2. $\operatorname{slrk}(f) = \min\{\operatorname{codim}(U) \mid U \subseteq \mathbb{C}^n, f|_U = 0\}$
3. The subset of polynomials of slice rank $\leq k$ closed.

Generic and maximal strength

Theorem (Harris)

A generic homogeneous polynomial of degree d in $n+1$ variables has slice rank

$$\min\left\{ r \in \mathbb{Z}_{\geq (n+1)/2} \;\middle|\; r(n+1-r) \geq \binom{d+n-r}{d} \right\}.$$

Theorem (B-Oneto)

The strength and slice rank of a homogeneous polynomial of degree d are generically equal for $d \leq 7$ and $d = 9$.

Theorem (Ballico-B-Oneto-Ventura)

The strength and slice rank of a homogeneous polynomial of degree d are generically equal for $d \geq 5$.

Generic and maximal strength

We consider

$$\{g_1 \cdot h_1 + \ldots + g_r \cdot h_r \mid \deg(g_i) = a_i, \deg(h_i) = d - a_i\}$$

inside $\mathbb{C}[x_1, \ldots, x_n]_d$. We want to know the dimension.

Terracini's Lemma

This dimension equals the dimension of $(g_1, h_1, \ldots, g_r, h_r)_d$ for generic generators.

Proposition

This dimension is at most

$$\binom{n+d}{d} - \mathrm{coeff}_d \left(\frac{\prod_{i=1}^r (1 - t^{a_i})(1 - t^{d-a_i})}{(1-t)^{n+1}} \right) + \binom{\ell_{d/2}}{2}$$

where $\ell_{d/2} := \#\{i \mid a_i = d/2\}$. Equality when all a_i equal to 1.

Generic and maximal strength

For fixed d, r, we want $F(a_1, \ldots, a_r) :=$

$$\mathrm{coeff}_d\left(\frac{\prod_{i=1}^r (1-t^{a_i})(1-t^{d-a_i})}{(1-t)^{n+1}}\right) - \binom{\ell_{d/2}}{2}$$

to be minimal when all a_i equal to 1.

Write $c_\ell(k_1, \ldots, k_n) := \mathrm{coeff}_\ell(P_{k_1} \cdots P_{k_n}) \geq 0$ where

$$P_k = 1 + t + \ldots + t^k$$

for $k \in \{0, 1, 2, \ldots\} \cup \{\infty\}$.

Proposition

We have

- $c_\ell(k_1, \ldots, k_n) = c_\ell(k_{\sigma(1)}, \ldots, k_{\sigma(n)})$ for all $\sigma \in S_n$
- $c_\ell(k_1, \ldots, k_n, 0) = c_\ell(k_1, \ldots, k_n)$
- $c_\ell(k, k_2, \ldots, k_n) \geq c_\ell(k', k_2, \ldots, k_n)$ for all $0 \leq k' \leq k \leq \infty$
- $c_{\ell+1}(k_1, \ldots, k_n) \geq c_\ell(k_1, \ldots, k_n)$ when $k_1 = \infty$

Generic and maximal strength

Proposition

We have

$$F(a_1,\dots,a_r)-F(a_1,\dots,a_{r-1},a_r-1)>0$$

when $a_r=\theta:=\max\{a_1,\dots,a_r\}>2$.

Proof.

Write $\ell_j=\#\{i\mid a_i=j\}$ and $m=n-\ell_1$. The difference equals

$$c_{d-\theta+1}(\infty^{n-r},d-2\theta,a_1-1,\dots,a_{r-1}-1)-\ell_{\theta-1}-(\ell_\theta-1)m.$$

We have

$$\begin{aligned}
&c_{d-\theta+1}(\infty^{n-r},d-2\theta,a_1-1,\dots,a_{r-1}-1)\\
\geq\, &c_{\theta+1}(\infty^{n-r},0^{\ell_1+1},1^{r-\ell_1-\ell_\theta},(\theta-1)^{\ell_\theta-1})\\
\geq\, &c_{\theta+1}(\infty,1^{m-\ell_\theta-1},(\theta-1)^{\ell_\theta-1})=\operatorname{coeff}_{\theta+1}(P_\infty P_1^{m-\ell_\theta-1}P_{\theta-1}^{\ell_\theta-1})\\
=\, &\operatorname{coeff}_{\theta+1}(P_\infty^{\ell_\theta}P_1^{m-\ell_\theta-1}(1-t^\theta)^{\ell_\theta-1})\\
=\, &\operatorname{coeff}_{\theta+1}(P_\infty^{\ell_\theta}P_1^{m-\ell_\theta-1})-(\ell_\theta-1)(m-1)\\
\geq\, &\operatorname{coeff}_4(P_\infty^{\ell_\theta}P_1^{m-\ell_\theta-1})-(\ell_\theta-1)(m-1)
\end{aligned}$$

□

Strength of polynomials

Q: How do you compute the strength of a polynomial?

Q: Is there an algorithm that computes best low-strength approximations of a polynomial?

Q: What is the highest possible strength of a limit of strength $\leq k$ polynomials?

Thanks for your attention!

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