Strength of infinite Polynomials

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Definition

The *strength* of a homogeneous polynomial f of degree $d \geq 2$ is the minimal r such that

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

where $\deg(g_i), \deg(h_i) < d$.

Example

What is the strength of $f = x^2 + y^2 + z^2$?

- We have $\operatorname{str}(f) \leq 3$ since $f = x \cdot x + y \cdot y + z \cdot z$.
- We have $\operatorname{str}(f) \neq 0$ since $f \neq 0$.
- We have $\operatorname{str}(f) \neq 1$ since f is not reducible.
- $\operatorname{str}_{\mathbb{C}}(f) = 2$ as $f = (x + iy) \cdot (x - iy) + z \cdot z$.
- $\operatorname{str}_{\mathbb{R}}(f) = 3$ as $f = g_1 h_1 + g_2 h_2 \Rightarrow \{g_1, g_2 = 0\} \subseteq \{f = 0\}$.

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Remark

Any $f \in \mathbb{C}[x_1, \dots, x_n]_d$ of degree $d \geq 2$ can be written as

$$f = x_1 \cdot h_1 + x_2 \cdot h_2 + \ldots + x_n \cdot h_n$$

and hence has strength $\leq n$.

Theorem (Harris)

For $d \geq 3$, any polynomial in $\mathbb{C}[x_1, \dots, x_n]_d$ can be written as

$$\ell_1 h_1 + \ldots + \ell_r h_r$$

with $\ell_1, \dots, \ell_r$ linear, where $r \approx n - \sqrt[d-1]{d!n}$ is minimal such that

$$r \geq \tfrac{1}{n-r}\tbinom{n-r+d-1}{d}.$$

Theorem (Ballico-B-Oneto-Ventura)

A generic polynomial in $\mathbb{C}[x_1, \dots, x_n]_d$ has strength r.

Consider $f = x_1^d + \ldots + x_n^d$.

We have

$$f = \begin{cases} (x_1^d + x_2^d) + \ldots + (x_{2k-1}^d + x_{2k}^d) & \text{if } n = 2k \\ (x_1^d + x_2^d) + \ldots + (x_{2k-1}^d + x_{2k}^d) + x_{2k+1}^d & \text{if } n = 2k+1 \end{cases}$$

and so $\mathrm{str}(f) \leq \lceil n/2 \rceil$.

We have

$$\begin{aligned} \mathrm{Sing}(f) &= \{\partial f/\partial x_1 = \ldots = \partial f/\partial x_n = 0\} \\ &= \{d x_1^{d-1} = \ldots = d x_n^{d-1} = 0\} = \{(0, \ldots, 0)\} \subseteq \mathbb{C}^n \end{aligned}$$

If $f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$, then

$$\{g_1, h_1, \ldots, g_r, h_r = 0\} \subseteq \mathrm{Sing}(f)$$

So $\mathrm{str}(f) \geq \lceil n/2 \rceil$.

Why care about strength?

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Universality

Let $f \in K[x_1, \dots, x_n]_d$ and $\ell_1, \dots, \ell_n$ be linear forms in $y_1, \dots, y_m$. The polynomial

$$f(\ell_1, \dots, \ell_n) \in K[y_1, \dots, y_m]_d$$

is a coordinate transform of f.

Let $\mathcal{P}$ be a property of degree-d polynomials such that

$$f \text{ has } \mathcal{P} \Rightarrow \text{every coordinate transform of } f \text{ has } \mathcal{P}$$

Example

$\mathcal{P} =$"is a limit of strength-r polynomials over $\overline{K}$"

Theorem (Kazhdan-Ziegler, B-Danelon-Draisma-Eggermont)

Either all f have $\mathcal{P}$ or there exists a $k \geq 0$ such that

$$f \text{ has } \mathcal{P} \Rightarrow \mathrm{str}(f) \leq k$$

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Adage

Infinite-strength polynomials behave simpler than high-strength polynomials.

Definition

The *collective strength* of a collection $(f_1, \ldots, f_k)$ is the minimal strength of its (nontrivial) linear combinations.

Theorem (Ananyan-Hochster)

For every collection $(d_1, \ldots, d_k)$ of degrees ≥ 1, there exists a constant C such that if $\deg(f_i) = d_i$ for all $i \in \{1, \ldots, k\}$ and $\mathrm{str}(f_1, \ldots, f_k) \geq C$, then $(f_1, \ldots, f_k)$ forms a regular sequence.

Polynomial series

For $d \geq 1$, we define

$$S_\infty^d = \left\{ \sum_{1 \leq i_1 \leq \cdots \leq i_d} a_{i_1 \cdots i_d} x_{i_1} \cdots x_{i_d} \;\middle|\; a_{i_1 \cdots i_d} \in \mathbb{C} \right\}$$

to be the set of degree-d *polynomial series*.

Now $S_\infty = \bigoplus_{d \geq 0} S_\infty^d \supsetneq \mathbb{C}[x_1, x_2, \ldots]$ is a ring.

Theorem (Erman-Sam-Snowden)

The ring S_∞ is a polynomial ring.

Idea of the proof

For every degree $d \geq 1$, take a basis of S_∞^d modulo finite strength.
The basis elements are the variables.

Polynomial series

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Example

The polynomial series $x_1^d + x_2^d + \ldots$ has infinite strength, since

$$\mathrm{str}(x_1^d + x_2^d + \ldots) \geq \mathrm{str}(x_1^d + x_2^d + \ldots + x_n^d) = \lceil n/2 \rceil$$

for all $n \geq 1$.

Example

The tuple

$$(x_1^d + x_5^d + \ldots\ , x_2^d + x_6^d + \ldots\ , x_3^d + x_7^d + \ldots\ , x_4^d + x_8^d + \ldots)$$

has infinite collective strength, since any nontrivial linear combination looks like the previous example.

Is bounded strength closed?

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Theorem

For every $k \geq 0$, the set $\{A \in \mathbb{C}^{n \times m} \mid \mathrm{rk}(A) \leq k\}$ is closed.

Question

What about $\{f \in \mathbb{C}[x_1, \ldots, x_n]_d \mid \mathrm{str}(f) \leq k\}$?

For $k = 1$, yes.

For $k = 2$, probably.

For $d = 2$, yes. (rank of symmetric matrices)

For $d = 3$, yes. (slice rank of polynomials)

Theorem (Ballico-B-Oneto-Ventura)

The set $\{f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \mathrm{str}(f) \leq 3\}$ is not closed for $n \gg 0$.

We have

$$\begin{aligned} 0 &= a^2 - b^2 - (a+b)(a-b) \\ &= x^2 \cdot y^2 - u^2 \cdot v^2 - (xy+uv) \cdot (xy-uv) \end{aligned}$$

for $a = xy$ and $b = uv$.

Consider

$$\frac{1}{t}\Big((x^2+tg)(y^2+tf) - (u^2-tq)(v^2-tp) - (xy-uv)(xy+uv)\Big)$$

for f, g, q, p of degree 2. For $t \to 0$, we get

$$x^2f + y^2g + u^2p + v^2q$$

Idea

Choose f, g, p, q such that $\mathrm{str}(x^2f + y^2g + u^2p + v^2q) = 4$.

How?

Let (f, g, p, q) have infinite collective strength.

Proposition

We have

$$\mathrm{str}(x^2 f + y^2 g + u^2 p + v^2 q) = 4$$

when x, y, u, v and f, g, p, q are variables of degrees 1 and 2.

1/4 of the proof

We need to show, for example, that

$$x^2 f + y^2 g + u^2 p + v^2 q \neq g_1 \cdot h_1 + g_2 \cdot h_2 + g_3 \cdot h_3$$

for all $g_i, h_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_2$. Write

$$g_i = G_i + \hat{g}_i \text{ and } h_i = H_i + \hat{h}_i$$

where $G_i, H_i \in \mathbb{C}[f, g, p, q]$ and $\hat{g}_i, \hat{h}_i \in \mathbb{C}[x, y, u, v]$. Then

$$\langle G_1, H_1, G_2, H_2, G_3, H_3 \rangle = \langle f, g, p, q \rangle$$

Now set $x, y, u, v = 0$.

A counterexample in the infinite setting

We get

$$G_1 \cdot H_1 + G_2 \cdot H_2 + G_3 \cdot H_3 = 0$$

for linear forms G_i, H_i in f, g, p, q that span $\langle f, g, p, q \rangle$.

After relabelling, we may assume that either

$$\langle G_1, H_1, G_2, H_2 \rangle = \langle f, g, p, q \rangle \text{ or } \langle G_1, H_1, G_2, G_3 \rangle = \langle f, g, p, q \rangle$$

(a) Write $(G_1, H_1, G_2, H_2) = (X, Y, U, V)$. Then we get

$$XY + UV = -G_3 \cdot H_3$$

which cannot happen as $XY + UV$ is irreducible.

(b) Write $(G_1, H_1, G_2, G_3) = (X, Y, U, V)$. Then we get

$$XY + UH_2 + VH_3 = 0$$

which cannot happen when we set $U, V = 0$. □

A counterexample in the finite setting

We now know that

$$x^2 \sum_{i=1}^{\infty} x_{4i+1}^2 + y^2 \sum_{i=1}^{\infty} x_{4i+2}^2 + u^2 \sum_{i=1}^{\infty} x_{4i+3}^2 + v^2 \sum_{i=1}^{\infty} x_{4i+4}^2$$

has strength 4.

Theorem (Lang)

Let $\mathcal{C}$ be a countable collection of polynomial equations over an uncountable field. Suppose that every finite $\mathcal{F} \subseteq \mathcal{C}$ has a solution. Then $\mathcal{C}$ has a solution.

Corollary

The strength of

$$x^2 \sum_{i=1}^{n} x_{4i+1}^2 + y^2 \sum_{i=1}^{n} x_{4i+2}^2 + u^2 \sum_{i=1}^{n} x_{4i+3}^2 + v^2 \sum_{i=1}^{n} x_{4i+4}^2$$

equals 4 for $n \gg 0$.

What does a coordinate transform mean in the infinite setting?

Non-example

Take $f = x_1 + x_2 + \ldots$ and set $x_i \mapsto x_1$ for all $i \in \mathbb{N}$.

Definition

Let $f \in S^d_\infty$. Then a *coordinate transform* of f is

$$f(\ell_1, \ell_2, \ldots) \in S^d_\infty$$

where $\ell_1, \ell_2, \ldots$ are linear forms in $x_1, x_2, \ldots$ so that every variable x_i only appears in finitely many linear forms ℓ_j.

Why does this work?

Suppose you want to know the coefficient of $x_{i_1} \cdots x_{i_d}$ in $f(\ell_1, \ell_2, \ldots)$. Let k be such that $x_{i_1}, \ldots, x_{i_d}$ only appear in $\ell_1, \ldots, \ell_k$ and consider $f(\ell_1, \ldots, \ell_k, 0, \ldots)$.

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Question

What is the structure of the set of infinite-strength degree-d polynomials up to coordinate transformation?

Example

For $d = 1$, everything is equivalent to x_1.

Proposition

For $d = 2$, everything is equivalent to $x_1^2 + x_2^2 + \dots$

Proof.

Infinite degree-2 polynomials f are the same as infinite symmetric matrices A. A coordinate transform of A is $P^\top AP$ where P is an infinite matrix where every column has only finitely many nonzero entries. The polynomial f has infinite strength if and only if A has infinite rank. Now start diagonalizing... $\square$

We focus on degree $d = 3$.

Replace $x_1, x_2, \dots$ by different countable set of variables when convenient.

Example

Every partial derivative $\partial f / \partial x_i$ of

$$f = x_1^3 + x_2^3 + \dots$$

has finite strength. Same for all its coordinate transforms:

$$\frac{\partial}{\partial x_i} f(\ell_1, \ell_2, \dots) = \sum_{j=1}^{\infty} \frac{\partial f}{\partial x_j}(\ell_1, \ell_2, \dots) \cdot \frac{\partial \ell_j}{\partial x_i}$$

is in fact a finite sum. So

$$x(y_1^2 + y_2^2 + \dots) + z_1^3 + z_2^3 + \dots$$

is not a coordinate transform of f.

Residual rank

Definition

The *residual rank* of $f \in S^d_\infty$ is

$$\mathrm{rrk}(f) = \dim \mathrm{span} \left\{ \frac{\partial}{\partial x_i} f \text{ mod } F^{d-1} \,\middle|\, i \in \mathbb{N} \right\}$$

where $F^{d-1} \subseteq S^{d-1}_\infty$ is the subspace of finite-strength elements.

Theorem (B-Danelon-Snowden)

The map rrk is an isomorphism between the poset of infinite-strength degree-3 polynomials and $\{0\} \cup \mathbb{N} \cup \{\infty\}$.

Idea of the proof

If $r = \mathrm{rrk}(f) < \infty$, show that

$$f \simeq x_1(y_{11}^2 + y_{12}^2 + \dots) + \dots + x_r(y_{r1}^2 + y_{r2}^2 + \dots) + z_1^3 + z_2^3 + \dots$$

If $\mathrm{rrk}(f) = \infty$, show that

$$f \simeq x_1(y_{11}^2 + y_{12}^2 + \dots) + x_2(y_{21}^2 + y_{22}^2 + \dots) + \dots$$

Residual rank

Assume $r = \mathrm{rrk}(f) < \infty$. First put f in standard form:

$$f \simeq x_1 g_1 + \ldots + x_r g_r + h$$

with $(g_1, \ldots, g_r, h)$ of infinite collective strength and $\mathrm{rrk}(h) = 0$.

We can write

$$\frac{\partial f}{\partial x_i} \equiv c_{1i} g_1 + \ldots + c_{ri} g_r \text{ mod finite strength}$$

for some $g_1, \ldots, g_r \in S^2_\infty$ and $c_1, \ldots, c_r \in S^1_\infty$.

As $\mathrm{rrk}(f) = r$, we see that $(g_1, \ldots, g_r)$ has infinite collective strength and $c_1, \ldots, c_r$ are linearly independent. Using r variables substitutions, we can assume that $c_j = x_j$.

Now $h = f - (x_1 g_1 + \ldots + x_r g_r)$ has $\mathrm{str}(h) = \infty$ and $\mathrm{rrk}(h) = 0$.

Residual rank

Assume that f in standard form:

$$f = x_1 g_1 + \ldots + x_r g_r + h$$

with $(g_1, \ldots, g_r, h)$ of infinite collective strength and $\mathrm{rrk}(h) = 0$.

Goal

Show that $(g_1, \ldots, g_r, h)$ is equivalent to $(\hat{g}_1, \ldots, \hat{g}_r, \hat{h}) =$

$$(x_{r+1}^2 + x_{2r+2}^2 + \ldots,\ \ldots\ ,\ x_{2r}^2 + x_{3r+1}^2 + \ldots,\ x_{2r+1}^3 + x_{3r+2}^3 + \ldots)$$

Recursion

Assume that for $k \geq n$ we have chosen linear forms $\ell_1, \ldots, \ell_k$ in $x_1, \ldots, x_n$ such that

$$g_i(\ell_1, \ldots, \ell_k, 0, \ldots) = \hat{g}_i(x_1, \ldots, x_n, 0, \ldots), \quad i \in \{1, \ldots, r\},$$
$$h(\ell_1, \ldots, \ell_k, 0, \ldots) = \hat{h}(x_1, \ldots, x_n, 0, \ldots)$$

We want to increase n (and k).

Residual rank

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Take $\hat{g}_i^{(n)} := \hat{g}_i(x_1, \dots, x_n, 0, \dots)$ and $\hat{h}^{(n)} := \hat{h}(x_1, \dots, x_n, 0, \dots)$

We have

$$g_i(\ell_1, \dots, \ell_k, x_{k+1}, \dots) = \hat{g}_i^{(n)} + \sum_{j=1}^{n} x_j a_{ij} + \overline{g}_i$$

$$h(\ell_1, \dots, \ell_k, x_{k+1}, \dots) = \hat{h}^{(n)} + \sum_{j=1}^{n} x_j f_j + \sum_{1 \le u \le v \le n} x_u x_v b_{uv} + \overline{h}$$

where $a_{ij}, b_{uv}, \overline{g}_i, f_j, \overline{h}$ only use $x_{k+1}, x_{k+2}, \dots$

Now, we want to choose linear forms $\ell_{k+1}, \dots, \ell_{k'}$ in x_{n+1} with

$$\begin{aligned} a_{ij}(\ell_{k+1}, \dots, \ell_{k'}, 0, \dots) &= 0 \\ b_{uv}(\ell_{k+1}, \dots, \ell_{k'}, 0, \dots) &= 0 \\ \overline{g}_i(\ell_{k+1}, \dots, \ell_{k'}, 0, \dots) &= \lambda_{ij} \cdot x_{n+1}^2 \\ f_j(\ell_{k+1}, \dots, \ell_{k'}, 0, \dots) &= 0 \\ \overline{h}(\ell_{k+1}, \dots, \ell_{k'}, 0, \dots) &= \mu_i \cdot x_{n+1}^3 \end{aligned}$$

Residual rank

Take $\hat{g}_i^{(n)} := \hat{g}_i(x_1, \dots, x_n, 0, \dots)$ and $\hat{h}^{(n)} := \hat{h}(x_1, \dots, x_n, 0, \dots)$

We have

$$g_i(\ell_1, \dots, \ell_k, x_{k+1}, \dots) = \hat{g}_i^{(n)} + \sum_{j=1}^{n} x_j a_{ij} + \overline{g}_i$$

$$h(\ell_1, \dots, \ell_k, x_{k+1}, \dots) = \hat{h}^{(n)} + \sum_{j=1}^{n} x_j f_j + \sum_{1 \le u \le v \le n}^{n} x_u x_v b_{uv} + \overline{h}$$

where $a_{ij}, b_{uv}, \overline{g}_i, f_j, \overline{h}$ only use $x_{k+1}, x_{k+2}, \dots$

Now, we want to choose constants $c_{k+1}, \dots, c_{k'} \in \mathbb{C}$ with

$$\begin{aligned}
a_{ij}(c_{k+1}, \dots, c_{k'}, 0, \dots) &= 0 \\
b_{uv}(c_{k+1}, \dots, c_{k'}, 0, \dots) &= 0 \\
\overline{g}_i(c_{k+1}, \dots, c_{k'}, 0, \dots) &= \lambda_{ij} \\
f_j(c_{k+1}, \dots, c_{k'}, 0, \dots) &= 0 \\
\overline{h}(c_{k+1}, \dots, c_{k'}, 0, \dots) &= \mu_i
\end{aligned}$$

Question

What about degree 4?

Assume $r = \mathrm{rrk}(f) < \infty$. Then we can put f in standard form

$$f \simeq x_1 g_1 + \ldots + x_r g_r + h$$

with $(g_1, \ldots, g_r, h)$ of infinite collective strength and $\mathrm{rrk}(h) = 0$.

Definition

The *second residual rank* of $f \in S^d_\infty$ is

$$\mathrm{rrk}^{(2)}(f) = \dim \mathrm{span} \left\{ \frac{\partial^2}{\partial x_i \partial x_j} f \text{ mod } F^{d-2} \,\middle|\, i, j \in \mathbb{N} \right\}$$

where $F^{d-2} \subseteq S^{d-2}_\infty$ is the subspace of finite-strength elements.

Question

Can we seperate the second-order content from the part that has infinite strength?

Thank you for your attention!