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Strength and polynomial functors

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Infinite vectors and matrices

Let K be an algebraically closed field of characteristic 0.

Definition:
(1) An infinite vector is a map $v\colon\ \mathbb{N} \to K$.
(2) An infinite matrix is a map $A\colon\ \mathbb{N} \times \mathbb{N} \to K$.

We write $v(i) = v_i$, $A(i,j) = A_{ij}$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix}, \qquad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots \\ A_{21} & A_{22} & A_{23} & \dots \\ A_{31} & A_{32} & A_{33} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

The rank of infinite matrices

Definition: The rank of an infinite matrix $A \in K^{\mathbb{N} \times \mathbb{N}}$ is

$$\mathrm{rk}(A) := \sup\{\mathrm{rk}(B) \mid \text{finite submatrices } B \text{ of } A\} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$$

Examples:
(1) The ranks of the matrices

$$I_\infty = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g & \\ & I_\infty \end{pmatrix} \text{ for } g \in \mathrm{GL}_n$$

are ∞.
(2) For non-zero infinite vectors $v, w \in K^{\mathbb{N}}$, the infinite matrix vw^T given by $(vw^T)_{ij} = v_i w_j$ has rank 1.

The rank of infinite matrices

Proposition:
$A \in K^{\mathbb{N} \times \mathbb{N}}$ has rank $\leqslant k < \infty \Leftrightarrow A = \sum_{j=1}^{k} v_j w_j^T$ with $v_j, w_j \in K^{\mathbb{N}}$

Proof. The direction $\Leftarrow$ is easy.

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Proof. The direction $\Leftarrow$ is easy.

For $\Rightarrow$, assume for convenience that both A and its topleft $k \times k$ submatrix have rank k. Let $v_1, \ldots, v_k \in K^{\mathbb{N}}$ be the first k columns of A.

Goal: prove that every column of A is a linear combination of $v_1, \ldots, v_n$.

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Let v' be another column of A and take

$$V_n := \{(\lambda', \lambda_1, \ldots, \lambda_k) \in K^{k+1} \mid \forall i \leqslant n : \lambda' v'_i = \lambda_1 v_{1i} + \cdots + \lambda_k v_{ki}\}$$

We have $V_{n+1} \subseteq V_n$ and $V_n \neq 0$ for all $n \in \mathbb{N}$.

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We have $V_{n+1} \subseteq V_n$ and $V_n \neq 0$ for all $n \in \mathbb{N}$.
$\Rightarrow \bigcap_{n \in \mathbb{N}} V_n \neq 0$
Take $(1, \lambda_1, \dots, \lambda_k) \in \bigcap_{n \in \mathbb{N}} V_n$. Then $v' = \lambda_1 v_1 + \cdots + \lambda_k v_k$. $\square$

The Zariski topology on $K^{\mathbb{N} \times \mathbb{N}}$

Definition: A polynomial function on $K^{\mathbb{N} \times \mathbb{N}}$ sends a matrix A to a finite polynomial expression of its entries A_{ij}.

Example: $f(A) = A_{11}^3 A_{22} - A_{12} A_{21}$

Nonexample: $f(A) = A_{11}^2 + A_{22}^2 + A_{33}^3 + \dots$

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Definition: A subset of $K^{\mathbb{N}\times\mathbb{N}}$ is Zariski-closed when it is of the form

$$\{A \in K^{\mathbb{N}\times\mathbb{N}} \mid f(A) = 0 \text{ for all } f \in S\}$$

where S is a set of polynomial functions on $K^{\mathbb{N}\times\mathbb{N}}$.

Example: Take $k \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. Then the set

$$\{A \in K^{\mathbb{N}\times\mathbb{N}} \mid \mathrm{rk}(A) \leqslant k\}$$

is Zariski-closed.

The rank of infinite-by-infinite matrices

Fact: An $n \times m$ matrix A has rank $\min(n, m) \Leftrightarrow \overline{\mathrm{GL}_n \cdot A \cdot \mathrm{GL}_m} = K^{n \times m}$

Theorem: A matrix $A \in K^{\mathbb{N} \times \mathbb{N}}$ has rank $\infty \Leftrightarrow \overline{\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty} = K^{\mathbb{N} \times \mathbb{N}}$

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Proof. ($\Leftarrow$) If $\mathrm{rk}(A) = k < \infty$, then

$$\overline{\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty} \subseteq \{A \in K^{\mathbb{N} \times \mathbb{N}} \mid \mathrm{rk}(A) \leqslant k\} \subsetneq K^{\mathbb{N} \times \mathbb{N}}$$

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($\Rightarrow$) Suppose $\overline{\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty} \subsetneq K^{\mathbb{N} \times \mathbb{N}}$. Then $f(\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty) = 0$ for some nonzero polynomial function f.

The function f uses only finitely many entries.

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The function f uses only finitely many entries.
$\Rightarrow$ The rank of a particular finite submatrix has to be non-maximal for every element in $\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty$.
$\Rightarrow$ The rank of a particular finite submatrix has to be non-maximal for every permutation of A.
$\Rightarrow$ The rank of A must be finite. $\square$

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Let $A \in K^{\mathbb{N} \times \mathbb{N}}$ be an infinite matrix.

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Corollary: Precisely one of the following holds:
(1) $\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty$ is dense in $K^{\mathbb{N} \times \mathbb{N}}$.
(2) $A = \sum_{j=1}^{k} v_j w_j^T$ with $v_j, w_j \in K^{\mathbb{N}}$.

Remark: Similar statements hold for:

(1) Tuples of matrices (Draisma, Eggermont)
(2) Homogeneous polynomials (B, Draisma, Eggermont)
(3) Tensors (B, Draisma, Eggermont)

Similar statements

Definition: The rank of a tuple of matrices $(A_1, \ldots, A_k)$ is

$$\mathrm{rk}(A_1, \ldots, A_k) := \inf\{\mathrm{rk}(\lambda_1 A_1 + \cdots + \lambda_k A_k) \mid (\lambda_1 : \cdots : \lambda_k) \in \mathbb{P}^{k-1}\}$$

Definition: The strength of a homogeneous polynomial f of degree $d \geqslant 2$ is the minimal $k \leqslant \infty$ such that $f = g_1 h_1 + \cdots + g_k h_k$ with $\deg(g_i), \deg(h_i) < d$.

Definition: The flattening rank of a d-way tensor t is the minimal $k \leqslant \infty$ such that $t = f_1 + \cdots + f_k$ with each tensor f_i has some rank-1 flattening.

Why look at infinite objects?

Let $A \in K^{\mathbb{N} \times \mathbb{N}}$ be an infinite matrix.

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(1) $\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty$ is dense in $K^{\mathbb{N} \times \mathbb{N}}$.
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Let $X \subsetneq K^{\mathbb{N} \times \mathbb{N}}$ be a $(\mathrm{GL}_\infty \times \mathrm{GL}_\infty)$-stable Zariski-closed subset.
$\Rightarrow \mathrm{rk}(X) \leqslant k$ for some $k < \infty$.

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$\Rightarrow \mathrm{rk}(X) \leqslant k$ for some $k < \infty$.

Let $X_n \subseteq K^{\mathbb{N} \times \mathbb{N}}$ be the projection on $K^{n \times n}$.
$\Rightarrow \mathrm{rk}(B) \leqslant k$ for all $B \in X_n$.
$\Rightarrow$ Matrices $B \in X_n$ can always be expressed using $2k$ vectors.

Remark: The bound k does not depend on n.

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Definition: A category $\mathcal{C}$ has objects $C, D \in \mathcal{C}$, morphisms $C \to D$ and compositions. Taking compositions is associative and for every object $C \in \mathcal{C}$ there is an identity $\mathrm{id}_C \colon C \to C$.

Examples:
(0) The category Set. Objects are sets and morphisms are maps.
(1) The category Vec. Objects are finite-dimensional vector spaces and morphisms are linear maps.
(2) The category Top. Objects are topological spaces and morphisms are continious maps.
(3) For $k \in \mathbb{N}$, the category Vec^k. Objects are k-tuples $V = (V_1, \ldots, V_k)$ and morphisms are k-tuples $\ell = (\ell_1, \ldots, \ell_k)$.

Categories and functors

Let $\mathcal{C}, \mathcal{D}$ be categories.
Definition: A functor $F\colon \mathcal{C} \to \mathcal{D}$ assigns

- to every object $C \in \mathcal{C}$ an object $F(C) \in \mathcal{D}$
- to every morphism $\ell\colon C \to C'$ a morphism $F(\ell)\colon F(C) \to F(C')$

such that $F(\ell \circ \ell') = F(\ell) \circ F(\ell')$ and $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$.

Examples:
(0) The functor $\mathrm{For}\colon \mathrm{Vec} \to \mathrm{Set}$ with $\mathrm{For}(V) = V$ and $\mathrm{For}(\ell) = \ell$.
(1) The functor $\mathrm{Zar}\colon \mathrm{Vec} \to \mathrm{Top}$ with $\mathrm{Zar}(V) = V$ and $\mathrm{Zar}(\ell) = \ell$.
(2) For $k \in \mathbb{N}$, the functor $\Delta\colon \mathrm{Vec} \to \mathrm{Vec}^k$ with $\Delta(V) = (V, \ldots, V)$ and $\Delta(\ell) = (\ell, \ldots, \ell)$.

Polynomial functors as polynomials

Vec^k = category of k-tuples of finite-dimensional vector spaces.

Definition: A polynomial functor $P \colon \mathrm{Vec}^k \to \mathrm{Vec}$

(1) assigns a vector space $P(V) \in \mathrm{Vec}$ to every $V \in \mathrm{Vec}^k$

(2) assigns a polynomial map

$$\begin{array}{rcl} \mathrm{Mor}(V, W) & \to & \mathrm{Hom}(P(V), P(W)) \\ \ell & \mapsto & P(\ell) \end{array}$$

to every pair $(V, W) \in \mathrm{Vec}^k \times \mathrm{Vec}^k$

such that $P(\mathrm{id}_V) = \mathrm{id}_{P(V)}$ and $P(\ell_1 \circ \ell_2) = P(\ell_1) \circ P(\ell_2)$.

Examples: Take $U \in \mathrm{Vec}$ fixed and $i \in \{1, \dots, k\}$.
(1) Take $C_U(V) = U$ for all $V \in \mathrm{Vec}^k$ and $C_U(\ell) = \mathrm{id}_U$ for all ℓ.
(2) Take $T_i(V) = V_i$ for all $V \in \mathrm{Vec}^k$ and $C_U(\ell) = \ell_i$ for all ℓ.

Polynomial functors as polynomials

Let P, Q be polynomial functors.

Definition: Define the direct sum $P \oplus Q$ by:

$(P \oplus Q)(V) = P(V) \oplus Q(V)$ and $(P \oplus Q)(\ell)(v, w) = (P(\ell)(v), Q(\ell)(w))$

Definition: Define the tensor product $P \otimes Q$ by:

$(P \otimes Q)(V) = P(V) \otimes Q(V)$ and $(P \otimes Q)(\ell)(v \otimes w) = P(\ell)(v) \otimes Q(\ell)(w)$

Examples:

(1) $T \oplus T$ is the polynomial functor of 2-tuples of vectors.

(2) $T_1 \otimes T_2$ is the polynomial functor of matrices.

(3) $T_1 \otimes \cdots \otimes T_k$ is the polynomial functor of k-way tensors.

Polynomial functors as polynomials

Let P, Q be polynomial functors.

Definition: The functor Q is a subfunctor of P when $Q(V) \subseteq P(V)$.

Suppose that Q is a subfunctor of P.

Definition: Define the quotient P/Q by $(P/Q)(V) = P(V)/Q(V)$.

Examples:

(1) $T \otimes T$ has S^2 and $\bigwedge^2$ as subfunctors.

(2) $T^{\otimes k} := T \otimes \cdots \otimes T$ has S^d as subfunctor.

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Fact: Every polynomial functor can be obtained from the constants C_U and variables $T_1, \ldots, T_k$ using direct sums, tensor products, subfunctors and quotients.

Polynomial functors as topological spaces

Definition: A closed subset $X \subseteq P$ assigns a closed subset

$$X(V) \subseteq P(V)$$

to every $V \in \mathrm{Vec}^k$ such that $P(\ell)(X(V)) \subseteq X(W)$ for all $\ell\colon V \to W$.

Examples:

(1) A closed subset of C_U is a closed subset of U.

(2) $\{$linearly dependent tuples of vectors$\} \subseteq T \oplus \cdots \oplus T$.

(3) $\{$matrices of rank $\leqslant r\} \subseteq T_1 \otimes T_2$.

(4) $\{$tensors of rank $\leqslant r\} \subseteq T_1 \otimes \cdots \otimes T_k$.

(5) $\{$polynomials that are zero on a codim $\leqslant r$ subspace$\} \subseteq S^d$.

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Remark: For every $V \in \mathrm{Vec}^k$, we have the action

$$\begin{array}{rcl} \mathrm{GL}(V) := \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_k) & \to & \mathrm{GL}(P(V)) \\ \ell = (\ell_1, \ldots, \ell_k) & \mapsto & P(\ell) \end{array}$$

Morphisms between polynomial functors

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Let P, Q be polynomial functors.

Definition: A polynomial transformation $\alpha\colon Q \to P$ is a family

$$(\alpha_V\colon Q(V) \to P(V))_{V\in \mathrm{Vec}^k}$$

of polynomial maps such that

$$\begin{array}{ccc} Q(V) & \xrightarrow{\alpha_V} & P(V) \\ \downarrow{\scriptstyle Q(\ell)} & & \downarrow{\scriptstyle P(\ell)} \\ Q(W) & \xrightarrow{\alpha_W} & P(W) \end{array}$$

commutes for all $\ell\colon V \to W$.

Example: Take $P = T_1 \otimes T_2$ and $Q = T_1 \oplus T_1 \oplus T_2 \oplus T_2$. Then

$$\begin{aligned} \alpha_{(V,W)}\colon V \oplus V \oplus W \oplus W &\to V \otimes W \\ (v_1, v_2, w_1, w_2) &\mapsto v_1 \otimes w_1 + v_2 \otimes w_2 \end{aligned}$$

defines an polynomial transformation $\alpha\colon Q \to P$.

Main theorem

Let P, Q be polynomial functors. Write $Q \prec P$ when $Q_{(d)}$ is a quotient of $P_{(d)}$ where d is maximal with $Q_{(d)} \not\cong P_{(d)}$.

Dichotomy Theorem (B-Draisma-Eggermont-Snowden)
Let $X \subseteq P$ be a closed subset. Then $X = P$ or there are polynomial functors $Q_1, \ldots, Q_k \prec P$ and $\alpha_i \colon Q_i \to P$ such that $X \subseteq \bigcup_i \mathrm{im}(\alpha_i)$.

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Examples

- $\{$matrices of rank $\leqslant r\} = \{v_1 w_1^T + \cdots + v_r w_r^T \mid v_i, w_i$ vectors$\}$
- $\{$degree-d polynomials that are zero on a codim $\leqslant r$ subspace$\} = \{\ell_1 g_1 + \cdots + \ell_r g_r \mid \deg(\ell_i) = 1, \deg(g_i) = d-1\}$

Applications

Theorem (Draisma)

Every descending chain $P \supsetneq X_1 \supseteq X_2 \supseteq \ldots$ of closed subsets stabilizes.

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Proof. Using induction on P: take $Q_1, \ldots, Q_k \prec P$ and $\alpha_i \colon Q_i \to P$ such that $X_1 \subseteq \bigcup_i \mathrm{im}(\alpha_i)$ and pull back the chain of closed subsets along each α_i. The resulting chains all have to stabilize. □

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Theorem (B-Draisma-Eggermont-Snowden)
Let $X \subseteq Q$ be a constructible subset and let $\alpha \colon Q \to P$ be a morphism. Then $\alpha(X)$ is constructible.

More analogues from finite-dimensional affine algebraic geometry?

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Thank you for your attention!

References

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