Strength of Polynomials

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The X-rank

Let f be an element of a vector space V.
Let $X \subseteq V$ be a subset.

Definition

The X-*rank* of f is:

- 0 for $f = 0$.
- 1 for $f \in X \setminus \{0\}$.
- r, with r minimal such that $f = x_1 + \ldots + x_r$ where $x_i \in X$.
- ∞, when such an r does not exists.

Examples

- $K^{n\times m}$ with $X = \{vw^\top \mid v \in K^n, w \in K^m\}$.
- $K^{n_1} \otimes \cdots \otimes K^{n_d}$ with $X = \{v_1 \otimes \cdots \otimes v_d \mid v_i \in K^{n_i}\}$.
- $K^{\infty\times\infty}$ with $X = \{vw^\top \mid v \in K^\infty, w \in K^\infty\}$.
- $K[x_1, \ldots, x_n]_d$ with $X = \{g \cdot h \mid 0 < \deg(g), \deg(h) < d\}$.

The strength of polynomials

Definition

The *strength* of f is the minimal number r such that

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

with $g_1, h_1, \ldots, g_r, h_r$ homogeneous polynomials of degree $\leq d - 1$.

Example

What is the strength of $f = x^2 + y^2 + z^2$?

- We have $\mathrm{str}(f) \leq 3$ since $f = x \cdot x + y \cdot y + z \cdot z$.
- We have $\mathrm{str}(f) \neq 0$ since $f \neq 0$.
- We have $\mathrm{str}(f) \neq 1$ since f is not reducible.
- Note that $f = (x + iy) \cdot (x - iy) + z \cdot z$.

So $\mathrm{str}(f) = 2$ over $\mathbb{C}$ (but over $\mathbb{R}$ it would be 3).

Questions

How to compute the strength of a polynomial? Good bounds?

Why care about strength?

Reason 1 - Stillman's conjecture

The proof uses strength.

Reason 2 - Universality

Let $f \in K[x_1, \ldots, x_n]_d$ and $\ell_1, \ldots, \ell_n$ be linear forms in $y_1, \ldots, y_m$. The polynomial

$$f(\ell_1, \ldots, \ell_n) \in K[y_1, \ldots, y_m]_d$$

is a coordinate transformation of f.

Let $\mathcal{P}$ be a property of degree-d polynomials such that

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{every coordinate transformation of } f \text{ has } \mathcal{P}$$

Theorem (Kazhdan-Ziegler, B-Danelon-Draisma-Eggermont)

Either all f have $\mathcal{P}$ or there exists a $k \geq 0$ such that

$$f \text{ has } \mathcal{P} \Rightarrow \mathrm{str}(f) \leq k$$

$\{f \in \mathbb{C}[x_1, \ldots, x_n]_d \mid \mathrm{str}(f) \leq k\}$ Zariski-closed?

For $k = 1$, yes.
For $k = 2$, I don't know.
For $d = 2$, yes.
For $d = 3$, yes.

Theorem (Ballico-B-Oneto-Ventura)

The set $\{f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \mathrm{str}(f) \leq 3\}$ is not closed for $n \gg 0$.

$$^1/_t(x^2+tg)(y^2+tf)-{}^1/_t(u^2-tq)(v^2-tp)-{}^1/_t(xy-uv)(xy+uv)$$

$$\downarrow$$

$$x^2f + y^2g + u^2p + v^2q$$

Questions

Explicit example? Bound on n? $x^2a^2 + y^2b^2 + u^2c^2 + v^2d^2$?

Generic strength

Take $d \geq 3$ and $r \approx n - \sqrt[d-1]{d!n}$ minimal such that

$$r(n-r) \geq \binom{d+n-1-r}{d}.$$

Theorem (Harris)

A generic polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$ can be written as

$$\ell_1 h_1 + \ldots + \ell_r h_r$$

with $\ell_1, \ldots, \ell_r$ linear.

Theorem (Ballico-B-Oneto-Ventura)

This does not hold when r is smaller and/or not all ℓ_i linear.

Questions

How to find high-strength polynomials? Equations?

Strength of polynomials

Q: How to compute best low-strength approximations of a polynomial?

Q: What is the highest possible strength of a limit of strength $\leq k$ polynomials?

Thanks for your attention!