Euclidean distance degrees of orthogonally invariant varieties

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joint work with Jan Draisma

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Euclidean distance degree of a variety

Let X be a closed subvariety of $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$ be a point.

Problem: Find the point on X closest to v.

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Complexify + v sufficiently general $\rightsquigarrow$ The answer is the ED degree.

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Then there exists a $d \in \mathbb{Z}_{\geqslant 0}$ such that

$$\left\{ x \in X^{\text{reg}} \,\middle|\, v - x \perp T_x X \right\}$$

has size d for sufficiently general $v \in V$.

We call this d the ED degree of X in V.

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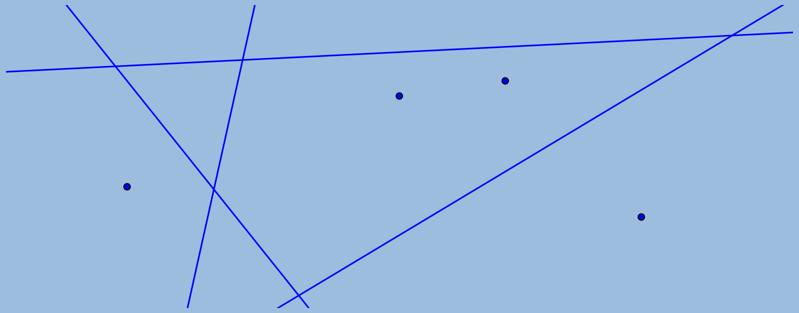
Remark: We may replace X^{reg} by any dense open subset U of X.

Example: points and lines



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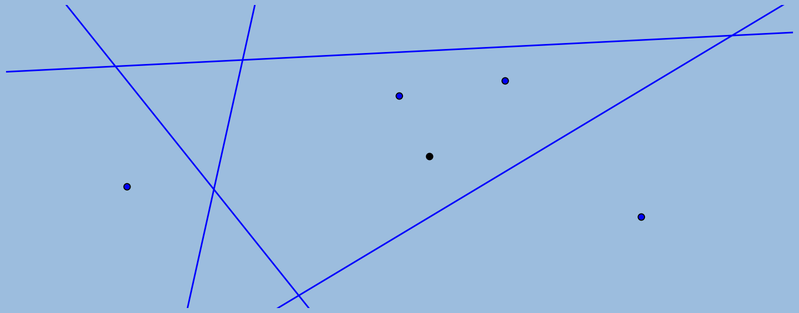

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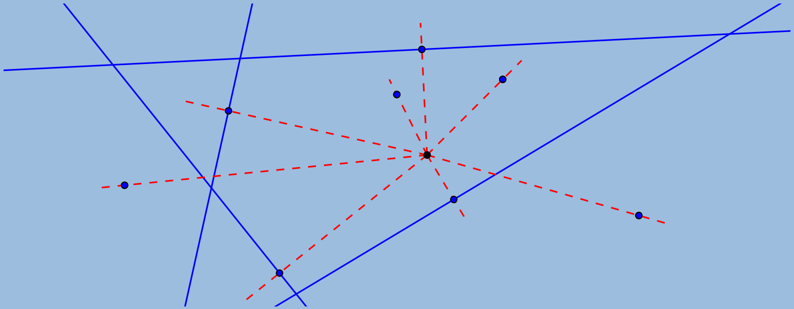

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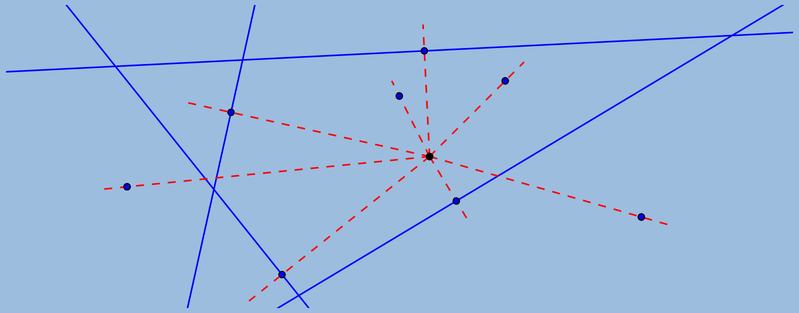

The ED degree equals the number of components.
(ignoring lines of the form $x \pm iy = a$)

Proposition

The ED degree of X is the sum of the ED degrees of its components.

Euclidean distance degree of a variety

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For points $x \in X$ on just one X_i, we have $T_x X = T_x X_i$.

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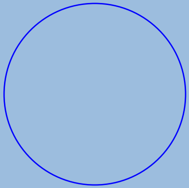
So

$$\{\text{critical points on } X\} = \dot{\bigcup_{i=1}^{n}} \{\text{critical points on } X_i\}$$

In particular, the sizes are equal. $\square$

Example: unit circle

$$x^2 + y^2 = 1$$



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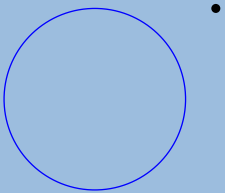

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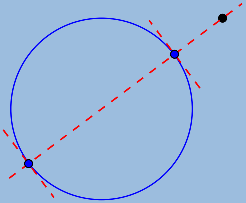

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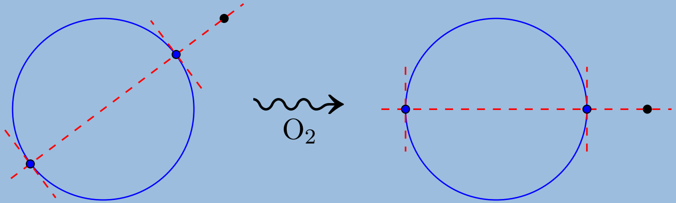

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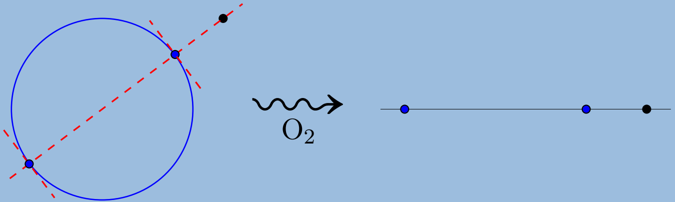

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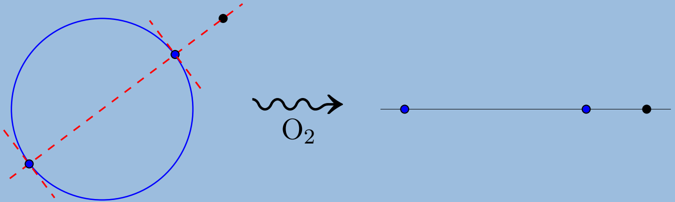

The ED degree is 2.

Orthogonally invariant matrix varieties

The group $\mathrm{O}(n) \times \mathrm{O}(m)$ acts on the space $\mathbb{C}^{n \times m}$ of $n \times m$ matrices. The bilinear form

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Let X be the closure in $\mathbb{C}^{n\times m}$ of a stable real subvariety of $\mathbb{R}^{n\times m}$ with smooth points. Let X_0 be the subset of X of diagonal matrices.

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We get the critical points by setting entries of v to 0.
$\rightsquigarrow$ the ED degree is $\binom{n}{k}$

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(1) $\mathrm{O}(n)X_0\mathrm{O}(m)$ *is dense in* X. *(Singular Value Decomposition)*

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(2) *For* $D \in \mathbb{C}^{n \times m}$ *a sufficiently general diagonal matrix, we have*

$$\mathbb{C}^{n \times m} = \{\text{diagonal matrices}\} \oplus T_D\left(\mathrm{O}(n) D \mathrm{O}(m)\right)$$

Orthogonally invariant varieties

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Theorem (B, Draisma, 2017)

Let $V_0 \subseteq V$ be a subspace and set $X_0 := X \cap V_0$. Assume that GX_0 is dense in X and that

$$V = V_0 \oplus T_{v_0} G v_0$$

for sufficiently general $v_0 \in V_0$.
Then the ED degree of X in V equals the ED degree of X_0 in V_0.

Proof for X, X_0 irreducible

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Let $v \in V$ and $v_0 \in V_0$ be sufficiently general. We want:

$$\# \left\{ x \in X^{\text{reg}} \middle| v - x \perp T_x X \right\} = \# \left\{ x \in X_0^{\text{reg}} \middle| v_0 - x \perp T_x X_0 \right\}$$

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  $\rightsquigarrow$ may assume $v = g \cdot \tilde{v}_0$ with $\tilde{v}_0 \in V_0$ s.g.

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Remark: First • uses that X_0 and X are irreducible. In general, we need to know more about how the components of X_0 and X are related.

Polar representations

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Theorem (B, Draisma, 2017)

The following are equivalent:

(1) V *has a subspace* V_0 *such that*

$$V = V_0 \oplus T_{v_0} G v_0$$

for sufficiently general $v_0 \in V_0$*.*

(2) V *is a stable polar representation.*

(3) $V_{\mathbb{R}}$ *is a polar representation.*

Definition (Dadok, Kac, 1985)

A complex representation V of an reductive algebraic group G is stable polar if there is a vector $v \in V$, whose orbit is maximal-dimensional and closed, such that the subspace

$$\{x \in V \mid T_x Gx \subseteq T_v Gv\}$$

has dimension $\dim(V//G)$.

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Definition (Dadok, 1985)

A real representation V of a compact Lie group K is polar if there is a vector $v \in V$, whose orbit is maximal-dimensional, such that for all $u \in (T_v Kv)^\perp$ we have $T_u Ku \subseteq T_v Kv$.

Dadok's classification

Dadok found all irreducible real polar representations of compact Lie groups. Complexification of Dadok's list:

G	V
G semisimple	$\mathfrak{g}$
$\mathrm{O}(n)$	$\mathbb{C}^n$
$\mathrm{O}(n)$	$\mathrm{Sym}^2(\mathbb{C}^n)$
$\mathrm{O}(n) \times \mathrm{O}(m)$	$\mathbb{C}^{n\times m}$
$\mathrm{Sp}(n)$	$\Lambda^2(\mathbb{C}^{2n})$
$\mathrm{Sp}(n) \times \mathrm{Sp}(m)$	$\mathbb{C}^{2n\times 2m}$
$\mathrm{SL}(V)$	$V \oplus V^*$
$\mathrm{GL}(V)$	$\mathrm{Sym}^2(V) \oplus \mathrm{Sym}^2(V)^*$
$\mathrm{GL}(V)$	$\Lambda^2(V) \oplus \Lambda^2(V)^*$
$\mathrm{Sp}(n)$	$\mathbb{C}^{2n} \oplus (\mathbb{C}^{2n})^*$
$\mathrm{GL}_n \times \mathrm{GL}_m$	$\mathbb{C}^{n\times m} \oplus (\mathbb{C}^{n\times m})^*$
SL_2	$\mathrm{Sym}^4(\mathbb{C}^2)$
$\vdots$	$\vdots$

Polar representations

The Weyl group

$$W = \{g \in G \mid gV_0 = V_0\}/\{g \in G \mid gv = v \text{ for all } v \in V_0\}$$

acts on V_0.

Polar representations

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Theorem (Dadok, Kac, 1985)

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Corollary

We have a one-to-one correspondence:

$$\begin{array}{rcl} \{Y \subseteq V_0 \mid Y \text{ is } W\text{-stable}\} & \leftrightarrow & \left\{X \subseteq V \,\middle|\, X = \overline{G(X \cap V_0)}\right\} \\ Y & \mapsto & \overline{GY} \\ X \cap V_0 & \leftarrow\!\shortmid & X \end{array}$$

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Thank you for your attention!

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