Polynomials of bounded strength

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joint work with Jan Draisma and Rob Eggermont

The strength of a polynomial

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$$x_1x_2x_5 + x_2x_3x_5 + x_3x_4x_5 + x_1x_5^2 + x_2x_5^2 + x_3x_5^2 - x_4x_5^2 +$$
$$x_1x_2x_6 + x_2x_3x_6 + x_3x_4x_6 - x_1x_6^2 - x_2x_6^2 - x_3x_6^2 + x_4x_6^2 +$$
$$x_1x_2x_7 + x_2x_3x_7 + x_3x_4x_7 - x_1x_7^2 - x_2x_7^2 - x_3x_7^2 + x_4x_7^2 +$$
$$x_1x_2x_8 + x_2x_3x_8 + x_3x_4x_8 + x_1x_8^2 + x_2x_8^2 + x_3x_8^2 - x_4x_8^2$$

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$$(x_1 + x_2 + x_3 - x_4)(x_5^2 - x_6^2 - x_7^2 + x_8^2) +$$
$$(x_1x_2 + x_2x_3 + x_3x_4)(x_5 + x_6 + x_7 + x_8)$$

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Definition

The strength of a homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ of degree $d \geq 2$ is the minimal $k \geq 0$ such that we can write

$$f = s_1 r_1 + \cdots + s_k r_k$$

with $s_1, \ldots, s_k, r_1, \ldots, r_k \in \mathbb{C}[x_1, \ldots, x_n]$ homogeneous polynomials of degree $< d$.

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Examples

- Reducible polynomials have strength ≤ 1.
- The polynomial $y^2 z - (x^3 + xz^2 + z^3)$ has strength 2.
- The polynomial $x_1^2 + \cdots + x_n^2$ has strength $\lceil n/2 \rceil$.
- Every polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]_{(d)}$ has strength $\leq n$.

The strength of a polynomial

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Proposition

For every symmetric matrix $A \in \mathbb{C}^{n\times n}$, the polynomial

$$f = (x_1 \ \dots \ x_n) A (x_1 \ \dots \ x_n)^T$$

has strength $\lceil \mathrm{rk}(A)/2 \rceil$.

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Remark

$f(x) = s_1(x)r_1(x) + \cdots + s_k(x)r_k(x)$ and $y_1, \dots, y_n$ are linear forms
$\Rightarrow f(y) = s_1(y)r_1(y) + \cdots + s_k(y)r_k(y)$ has strength $\leq k$

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Proof.

Change coordinates so that $f = x_1^2 + \cdots + x_r^2$ with $r = \mathrm{rk}(A)$.
$(\leq)$ Use: $x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2)$
$(\geq)$ Use: $2s_1r_1 = (x_1 \ \dots \ x_n)(vw^T + wv^T)(x_1 \ \dots \ x_n)^T$ $\square$

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The Main Theorem

Theorem (B, Draisma, Eggermont)

Let X *be a closed subfunctor of the functor* $S^d\colon \mathrm{Vec} \to \mathrm{Top}$*. Suppose that* $X(U) \neq S^d U$ *for some* $U \in \mathrm{Vec}$*. Then there is a constant* $k \in \mathbb{N}$ *such that the strength of any polynomial* $f \in X(V)$ *is at most* k*.*

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A closed subfunctor X of S^d assigns to every finite-dimensional vector space V a Zariski-closed subset $X(V)$ of $S^d V$ such that

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Example ($d = 2,\ S^2\,\mathbb{C}^n \cong \{\text{symmetric } A \in \mathbb{C}^{n\times n}\}$)

Take $X(\mathbb{C}^n) = \{\text{symmetric } n \times n \text{ matrices of rank } \leq r\}$. Then

- $X(\mathbb{C}^n)$ is the zero set of some subdeterminants
- for any $P \in \mathbb{C}^{n\times m}$ and any $A \in X(\mathbb{C}^n)$, we have $P^T AP \in X(\mathbb{C}^m)$
- $X(\mathbb{C}^{r+1}) \neq \{\text{symmetric } (r+1) \times (r+1) \text{ matrices}\}$

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The constant k depends only on d and $\dim U$ (and not on V).

Example

Take $d = 2$ and $X(V) = \{v \cdot v \mid v \in V\}$ for all $V \in \mathrm{Vec}$.
$\Rightarrow X(\mathbb{C}^2) = \{ax^2 + bxy + cy^2 \mid b^2 - 4ac = 0\} \subsetneq S^2\,\mathbb{C}^2$

The proof

Write $X_n = X(\mathbb{C}^n)$

$\{\ell(x_1, \ldots, x_n)^2\}$

The proof

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Write $X_n = X(\mathbb{C}^n)$

Fix m such that $X_m \neq \mathbb{C}[x_1, \ldots, x_m]_{(d)}$

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We do induction on the degree of the equation P

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$P \neq$ const $\Rightarrow P$ has a partial derivative $Q \neq 0$

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Take $Y_n = \{f \in X_n \mid \forall L\colon \mathbb{C}^m \to \mathbb{C}^n$ holds $Q(f \circ L) = 0\}$

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Consider polynomials in $X_n \setminus Y_n$

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Take $n = m + k$ and $y_i = x_{m+i}$. Then

$$f = g(y_1, \ldots, y_k) + \sum_{(i_1,\ldots,i_m)\neq 0} x_1^{i_1} \ldots x_m^{i_m} h_{i_1,\ldots,i_m}(y_1, \ldots, y_k)$$

has strength at most m plus the strength of g.

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$$f = ax_1^2 + bx_1x_2 + cx_2^2 + x_1\ell_1(y_1, \ldots, y_k) + x_2\ell_2(y_1, \ldots, y_k) + g(y_1, \ldots, y_k)$$

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$f \in X_n \Rightarrow f = a\left(x_1 + \frac{b}{2a}x_2 + \frac{1}{2a}\ell_1(y_1, \ldots, y_k)\right)^2$

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So $g(y_1,\dots,y_k)$ is a polynomial in $\ell_1(y_1,\dots,y_k)$ and $\ell_2(y_1,\dots,y_k)$

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So $g(y_1,\ldots,y_k)$ is a polynomial in $\ell_1(y_1,\ldots,y_k)$ and $\ell_2(y_1,\ldots,y_k)$

$\Rightarrow$ the strength of g is at most 2

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If $f \in X_n$ and $n \leq m$, then $f = x_1 r_1 + \cdots + x_n r_n$ has strength $\leq m$.

Take $n = m + k$ and $y_i = x_{m+i}$. Then

$$f = g(y_1, \ldots, y_k) + \sum_{(i_1,\ldots,i_m)\neq 0} x_1^{i_1} \ldots x_m^{i_m} h_{i_1,\ldots,i_m}(y_1, \ldots, y_k)$$

has strength at most m plus the strength of g.

$f \notin Y_n \Rightarrow Q(f \circ L) \neq 0$ for some $L\colon\ \mathbb{C}^m \to \mathbb{C}^n$

$f \in X_n \Rightarrow P(f \circ L') = 0$ for all $L'\colon\ \mathbb{C}^m \to \mathbb{C}^n$

$\rightsquigarrow g(y_1, \ldots, y_k)$ is a polynomial in the $h_{i_1,\ldots,i_m}$ $\square$

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Thank you for your attention!

Questions

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- Is the set $\{f \in \mathbb{C}[x_1, \ldots, x_n]_{(d)} \mid \text{strength}(f) \leq k\}$ Zariski-closed?
- What is the strength of a generic polynomial in $\mathbb{C}[x_1, \ldots, x_n]_{(d)}$?
- How do you calculate the strength of a polynomial?

For $d = 2$:

- Yes
- $\lceil n/2 \rceil$
- Compute the rank of the corresponding matrix

For $d = 3$:

- Yes
- $\min\left\{\ell \geq \frac{n}{2} \;\middle|\; \ell \in \mathbb{Z} \text{ and } \binom{d-\ell+n-1}{d} \leq \ell(n-\ell)\right\}$
- Find biggest subspace $U \subseteq V$ with $f(U) = 0$

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