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Polynomials of bounded strength

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The strength of a polynomial

$$x_1x_2x_5 + x_2x_3x_5 + x_3x_4x_5 + x_1x_5^2 + x_2x_5^2 + x_3x_5^2 - x_4x_5^2 +$$
$$x_1x_2x_6 + x_2x_3x_6 + x_3x_4x_6 - x_1x_6^2 - x_2x_6^2 - x_3x_6^2 + x_4x_6^2 +$$
$$x_1x_2x_7 + x_2x_3x_7 + x_3x_4x_7 - x_1x_7^2 - x_2x_7^2 - x_3x_7^2 + x_4x_7^2 +$$
$$x_1x_2x_8 + x_2x_3x_8 + x_3x_4x_8 + x_1x_8^2 + x_2x_8^2 + x_3x_8^2 - x_4x_8^2$$
$$=$$
$$(x_1 + x_2 + x_3 - x_4)(x_5^2 - x_6^2 - x_7^2 + x_8^2) +$$
$$(x_1x_2 + x_2x_3 + x_3x_4)(x_5 + x_6 + x_7 + x_8)$$

The strength of a polynomial

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Definition

The strength of a homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ of degree $d \geq 2$ is the minimal $k \geq 0$ such that we can write

$$f = s_1 r_1 + \cdots + s_k r_k$$

with $s_1, \ldots, s_k, r_1, \ldots, r_k \in \mathbb{C}[x_1, \ldots, x_n]$ homogeneous polynomials of degree $< d$.

Examples

- Reducible polynomials have strength ≤ 1.
- The polynomial $y^2 z - (x^3 + xz^2 + z^3)$ has strength 2.
- The polynomial $x_1^2 + \cdots + x_n^2$ has strength $\lceil n/2 \rceil$.
- Every polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]_{(d)}$ has strength $\leq n$.

The strength of a polynomial

Proposition

For every symmetric matrix $A \in \mathbb{C}^{n\times n}$, the polynomial

$$f = (x_1 \;\dots\; x_n) A (x_1 \;\dots\; x_n)^T$$

has strength $\lceil \mathrm{rk}(A)/2 \rceil$.

Remark

$f(x) = s_1(x)r_1(x) + \cdots + s_k(x)r_k(x)$ and $y_1, \dots, y_n$ are linear forms
$\Rightarrow f(y) = s_1(y)r_1(y) + \cdots + s_k(y)r_k(y)$ has strength $\leq k$

Proof.

Change coordinates so that $f = x_1^2 + \cdots + x_r^2$ with $r = \mathrm{rk}(A)$.
$(\leq)$ Use: $x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2)$
$(\geq)$ Use: $2s_1r_1 = (x_1 \;\dots\; x_n)(vw^T + wv^T)(x_1 \;\dots\; x_n)^T$ $\square$

The strength of a polynomial

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Vec-closed subfunctors of S^d

Theorem

Suppose that we have "nicely interacting" Zariski-closed subsets

$$X_n \subseteq \mathbb{C}[x_1, \ldots, x_n]_{(d)}, \quad n \in \mathbb{N}$$

such that $X_{n_0} \neq \mathbb{C}[x_1, \ldots, x_{n_0}]_{(d)}$ for some n_0. Then there is a constant $k \in \mathbb{N}$ such that the strength of any $f \in X_n$ is at most k.

Example ("nicely interacting" matrices)

Take $X_n = \{\text{symmmetric matrices of rank} \leq r\} \subseteq \mathbb{C}^{n \times n}$. Then

- X_n is the zero set of some subdeterminants
- for any $P \in \mathbb{C}^{n \times m}$ and any $A \in X_n$, we have $P^T AP \in X_m$
- $X_{r+1} \neq \{\text{symmetric } (r+1) \times (r+1) \text{ matrices}\}$

Theorem

Suppose that we have Zariski-closed subsets

$$X_n \subseteq \mathbb{C}[x_1, \dots, x_n]_{(d)}, \quad n \in \mathbb{N}$$

with

- $X_{n_0} \neq \mathbb{C}[x_1, \dots, x_{n_0}]_{(d)}$ *for some* n_0
- $f \circ L \in X_m$ *for each linear map* $L\colon \mathbb{C}^m \to \mathbb{C}^n$ *and each* $f \in X_n$

Then there is an $k \in \mathbb{N}$ *such that the strength of any* $f \in X_n$ *is at most* k*.*

(*) knowing $n_0 \Rightarrow$ upperbound on k.

The proof

Fix m such that $X_m \neq \mathbb{C}[x_1, \ldots, x_m]_{(d)}$
$\Rightarrow \exists P \neq 0$ so that $P(f) = 0$ for all $f \in X_m$

We do induction on the degree of the equation P.

$P \neq 0 \Rightarrow P$ has a partial derivative $Q \neq 0$ (with lower degree)

Take $Y_n = \{f \in X_n \mid \forall L\colon\ \mathbb{C}^n \to \mathbb{C}^m \text{ holds } Q(f \circ L) = 0\}$

$\deg(Q) < \deg(P) \Rightarrow$ done for polynomials $f \in Y_n$.

Consider polynomials in $X_n \setminus Y_n$.

$$Y_n = \{f \in X_n \mid \forall L\colon\ \mathbb{C}^n \to \mathbb{C}^m : Q(f \circ L) = 0\}$$

If $f \in X_n$ and $n \leq m$, then $f = x_1 r_1 + \cdots + x_n r_n$ has strength $\leq m$.

Take $n = m + k$ and $y_i = x_{m+i}$. Then

$$f = g(y_1, \ldots, y_k) + \sum_{(i_1,\ldots,i_m) \neq 0} x_1^{i_1} \ldots x_m^{i_m} h_{i_1,\ldots,i_m}(y_1, \ldots, y_k)$$

has strength at most m plus the strength of g.

$f \notin Y_n \Rightarrow Q(f \circ L) \neq 0$ for some $L\colon\ \mathbb{C}^n \to \mathbb{C}^m$

$f \in X_n \Rightarrow P(f \circ L') = 0$ for all $L'\colon\ \mathbb{C}^n \to \mathbb{C}^m$
$\Rightarrow Q(f \circ L) \cdot g$ is a polynomial in the $h_{i_1,\ldots,i_m}$ $\square$

Questions

Is the set $\{f \in \mathbb{C}[x_1, \ldots, x_n]_{(d)} \mid \text{strength}(f) < k\}$ Zariski-closed?

- yes for $d = 2$

What is the strength of a generic polynomial in $\mathbb{C}[x_1, \ldots, x_n]_{(d)}$?

- $\lceil n/2 \rceil$ for $d = 2$

How do you calculate the strength of a polynomial?

- calculate rank of the corresponding matrix for $d = 2$

References

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[b] UNIVERSITÄT BERN

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