Jordan Algebras of Symmetric Matrices

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Spaces of Symmetric Matrices

Let $\mathbb{S}^n$ be the space of symmetric $n \times n$ matrices over $\mathbb{C}$.

The Grassmannian $\mathrm{Gr}(m, \mathbb{S}^n)$ consists of m-dimensional $\mathcal{L} \subseteq \mathbb{S}^n$.

We here consider *regular* subspaces $\mathcal{L} \subseteq \mathbb{S}^n$:

$$\mathcal{L}_{\mathrm{inv}} := \{X \in \mathcal{L} \mid \det(X) \neq 0\} \neq \emptyset$$

Definition

The *reciprocal variety* $\mathcal{L}^{-1}$ is $\overline{\{X^{-1} \mid X \in \mathcal{L}_{\mathrm{inv}}\}} \subseteq \mathbb{S}^n$.

Goal

Understand the $\mathcal{L}$ where the variety $\mathcal{L}^{-1}$ is a linear space in $\mathbb{S}^n$.

The motivation for this goal arises in optimization (semidefinite programming) and in statistics (Gaussian models that are linear in covariance matrices and concentration matrices).

Spaces of Symmetric Matrices

Examples

$$\begin{pmatrix} x & y & & w \\ y & z & -w & \\ & -w & x & y \\ w & & y & z \end{pmatrix}, \quad \begin{pmatrix} x & & & & \\ & x & & & \\ & & y & & \\ & & & y & \\ & & & & z \end{pmatrix}, \quad \begin{pmatrix} x & & & & \\ & z & & & y \\ & & z & y & \\ & & y & & \\ & y & & & \end{pmatrix},$$

$$\begin{pmatrix} z & y & x & & & \\ y & x & & & & \\ x & & & & & \\ & & & y & x & \\ & & & x & & \\ & & & & & x \end{pmatrix}, \quad \begin{pmatrix} & y & z & & & x \\ y & & & & x & \\ z & & & x & & \\ & & x & & & \\ & x & & & & \\ x & & & & & \end{pmatrix}$$

Jordan Spaces of Symmetric Matrices

Theorem (B-Eisenmann-Sturmfels 2020, Jensen 1988)

For $\mathcal{L} \in \mathrm{Gr}(m, \mathbb{S}^n)$ and $U \in \mathcal{L}_{\mathrm{inv}}$, the following are equivalent:

(a) The reciprocal variety $\mathcal{L}^{-1}$ is also a linear space in $\mathbb{S}^n$.

(b) $\mathcal{L}$ is a subalgebra of the Jordan algebra $(\mathbb{S}^n, \bullet_U)$.

(c) $\mathcal{L}^{-1}$ equals $\mathcal{L}$ up to congruence; namely $\mathcal{L}^{-1} = U^{-1}\,\mathcal{L}\,U^{-1}$.

We say that $\mathcal{L}$ is a *Jordan space* when these equivalent conditions are satisfied.

Definition

For $U \in \mathbb{S}^n_{\mathrm{inv}}$, we define an algebra structure on $\mathbb{S}^n$ by

$$X \bullet_U Y := (XU^{-1}Y + YU^{-1}X)/2$$

for all $X, Y \in \mathbb{S}^n$. This makes $\mathbb{S}^n$ into a (unital) Jordan algebra:

$$X^{\bullet 2} \bullet (X \bullet Y) = X \bullet (X^{\bullet 2} \bullet Y).$$

Jordan Spaces of Symmetric Matrices

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Remark 1

For congruent subspaces $\mathcal{L}$ and $\mathcal{L}' = P\,\mathcal{L}\,P^\top$:

$$\mathcal{L} \text{ is a Jordan space } \Leftrightarrow \mathcal{L}' \text{ is also a Jordan space}$$

Remark 2

All choices of unit U lead to isomorphic Jordan algebras $(\mathcal{L}, \bullet_U)$.

Projective spaces

Let V be a vector space.

Definition

The projective space

$$\mathbb{P}(V) := \{[v] \mid v \in V \setminus \{0\}\}$$

where $[v] = [w]$ when $v = \lambda w$ for some $\lambda \neq 0$.

A subvariety of $\mathbb{P}(V)$ is defined by homogeneous polynomials:

for some $d \geq 0$: $\quad f(\lambda v) = \lambda^d f(v)$ for all $\lambda \in \mathbb{C}$ and $v \in V$

Theorem

Projective spaces are complete. So projections of closed subsets $Y \subseteq X \times \mathbb{P}(V)$ to X are closed. In particular, images from projective spaces are closed.

Example

The image of the map

$$\begin{array}{rcl} \mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^n) & \to & \mathbb{P}(\mathbb{C}^n \times \mathbb{C}^n) \\ ([\lambda, \mu], [v]) & \mapsto & [\lambda v, \mu v] \end{array}$$

is closed.

In fact, it is the set of linearly dependent vectors in $\mathbb{C}^n \times \mathbb{C}^n$.
So a point $[(x_1, \ldots, x_n), (y_1, \ldots, y_n)]$ is in the image of the map if and only if the matrix

$$\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$$

has rank ≤ 1. The polynomials $x_i y_j - x_j y_i$ are homogeneous.

The variety of Jordan spaces

Let V be a vector space of dimension n.

Coordinate systems for the Grassmannian

A subspace $\mathcal{L} \in \mathrm{Gr}(m, V)$ can be represented by:

	primal	dual
Stiefel	$(H_1, \ldots, H_{n-m}) \in (V^*)^{n-m}$	$(X_1, \ldots, X_m) \in V^m$
Plücker	$H_1 \wedge \cdots \wedge H_{n-m} \in \bigwedge^{n-m} V^*$	$X_1 \wedge \cdots \wedge X_m \in \bigwedge^m V$

Here

$$\mathcal{L} = \{v \in V \mid H_1(v), \ldots, H_{n-m}(v) = 0\} = \mathrm{span}(X_1, \ldots, X_m)$$

Proposition

The subset $\mathrm{Jo}(m, \mathbb{S}^n)$ consisting of all Jordan spaces $\mathcal{L}$ is a subvariety of $\mathrm{Gr}(m, \mathbb{S}^n)$.

Proof

The subspace $\mathcal{L} \in \mathrm{Gr}(m, \mathbb{S}^n)$ is a Jordan space $\Leftrightarrow$

(b) $\mathcal{L}$ is a subalgebra of the Jordan algebra $(\mathbb{S}^n, \bullet_U)$.

for all $U \in \mathcal{L}_{\mathrm{inv}}$.

Let $X_1, \ldots, X_m$ be a basis of $\mathcal{L}$. (Dual Stiefel coordinates)

Then (b) for all $U \in \mathcal{L}_{\mathrm{inv}}$ $\Leftrightarrow$

$$X_1, \ldots, X_m, X_i \bullet_U X_j$$

are linearly dependent for all $1 \leq i \leq j \leq m$ and $U \in \mathcal{L}_{\mathrm{inv}}$

Proof

The subspace $\mathcal{L} \in \mathrm{Gr}(m, \mathbb{S}^n)$ is a Jordan space $\Leftrightarrow$

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Let $X_1, \ldots, X_m$ be a basis of $\mathcal{L}$. (Dual Stiefel coordinates)

Then (b) for all $U \in \mathcal{L}_{\mathrm{inv}} \Leftrightarrow$

$$X_1, \ldots, X_m, (X_i U^{-1} X_j + X_j U^{-1} X_i)/2$$

are linearly dependent for all $1 \leq i \leq j \leq m$ and $U \in \mathcal{L}_{\mathrm{inv}}$

Proof

The subspace $\mathcal{L} \in \mathrm{Gr}(m, \mathbb{S}^n)$ is a Jordan space $\Leftrightarrow$

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Then (b) for all $U \in \mathcal{L}_{\mathrm{inv}} \Leftrightarrow$

$$X_1, \ldots, X_m, (X_i \operatorname{adj}(U) X_j + X_j \operatorname{adj}(U) X_i)$$

are linearly dependent for all $1 \leq i \leq j \leq m$ and $U \in \mathcal{L}_{\mathrm{inv}}$

Proof

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Let $X_1, \ldots, X_m$ be a basis of $\mathcal{L}$. (Dual Stiefel coordinates)

Then (b) for all $U \in \mathcal{L}_{\mathrm{inv}} \Leftrightarrow$

$$X_1, \ldots, X_m, (X_i \operatorname{adj}(U) X_j + X_j \operatorname{adj}(U) X_i)$$

are linearly dependent for all $1 \leq i \leq j \leq m$ and

$$U = c_1 X_1 + \ldots + c_m X_m \in \mathcal{L}$$

for all $c_1, \ldots, c_m \in \mathbb{C}$. □

Jordan pencils, nets, webs, ...

We call elements of $\mathrm{Gr}(2, \mathbb{S}^n)$ *pencils* of symmetric matrices.

Congruence orbits of regular pencils are classified by Segre symbols.

Definition

Let $\mathcal{L} \in \mathrm{Gr}(2, \mathbb{S}^n)$ be a regular pencil. The Segre symbol σ of $\mathcal{L}$ is a multiset of partitions adding up to n.

Pick a basis X, Y of $\mathcal{L}$ with $Y \in \mathcal{L}_{\mathrm{inv}}$. Then the Segre symbol of $\mathcal{L}$ is given by sizes of Jordan blocks of XY^{-1}.

Examples

$$\begin{pmatrix} x & & & & \\ & x & & & \\ & & y & & \\ & & & y & \\ & & & & y \end{pmatrix} \text{ and } \begin{pmatrix} y & x & & & \\ x & & & & \\ & & y & & \\ & & & y & \\ & & & & y \end{pmatrix}$$

have Segre symbols $[(1,1),(1,1,1)]$ and $[(2),(1,1,1)]$.

Jordan pencils, nets, webs, ...

We know which Segre symbols correspond to Jordan pencils.

Theorem (Fevola-Mandelshtam-Sturmfels 2020)

A pencil is a Jordan space exactly when its Jordan symbol is of the form $\sigma = [(1,\dots,1),(1,\dots,1)]$ or $\sigma = [(2,\dots,2,1\dots,1)]$.

$$\begin{pmatrix} x & & & & & \\ & \ddots & & & & \\ & & x & & & \\ & & & y & & \\ & & & & \ddots & \\ & & & & & y \end{pmatrix} \quad \begin{pmatrix} y & x & & & & & & \\ x & & & & & & & \\ & & \ddots & & & & & \\ & & & y & x & & & \\ & & & x & & & & \\ & & & & & x & & \\ & & & & & & \ddots & \\ & & & & & & & x \end{pmatrix}$$

The irreducible components of $\mathrm{Jo}(2,\mathbb{S}^n)$ are the orbits closures of the diagonalizable pencils.

Jordan pencils, nets, webs, ...

We call elements of $\mathrm{Gr}(3, \mathbb{S}^n)$ *nets* of symmetric matrices.

For $n = 2$, we have $\mathrm{Gr}(3, \mathbb{S}^2) = \{\mathbb{S}^2\}$ and $\mathbb{S}^2$ is a Jordan net.

For $n = 3$, all Jordan nets are congruent to one of:

$$\begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \to \begin{pmatrix} x & & \\ & z & y \\ & y & \end{pmatrix} \to \begin{pmatrix} z & y & x \\ y & x & \\ x & & \end{pmatrix}$$

For $n = 4$, the diagram becomes more interesting.

Jordan nets of symmetric 4×4 matrices

codim 9

$$\begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}$$

codim 10

$$\begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix} \quad \begin{pmatrix} x & & & \\ & z & & y \\ & & y & \\ & y & & \end{pmatrix} \quad \begin{pmatrix} x & y & & \\ y & z & & \\ & & x & y \\ & & y & z \end{pmatrix}$$

codim 11

$$\begin{pmatrix} z & y & x & \\ y & x & & \\ x & & & \\ & & & x \end{pmatrix} \quad \begin{pmatrix} y & & & x \\ & z & x & \\ & x & & \\ x & & & \end{pmatrix} \quad \begin{pmatrix} & x & z & \\ x & & & \\ z & & & y \\ & & y & \end{pmatrix}$$

codim 12

$$\begin{pmatrix} z & y & & x \\ y & & x & \\ & x & & \\ x & & & \end{pmatrix}$$

Theorem

Every (unital) Jordan algebra $\mathcal{A}$ of dimension 3 over $\mathbb{C}$ is isomorphic to the Jordan algebra $\mathbb{C}\{U, X, Y\}$ with unit U, where the product is given by

$$
\begin{aligned}
1\ (\mathrm{a}):&\quad X^{\bullet 2} = X, Y^{\bullet 2} = Y \text{ and } X \bullet Y = 0,\\
(\mathrm{b}):&\quad X^{\bullet 2} = U,\ Y^{\bullet 2} = U \text{ and } X \bullet Y = 0,\\
2\ (\mathrm{a}):&\quad X^{\bullet 2} = X, Y^{\bullet 2} = 0 \text{ and } X \bullet Y = 0,\\
(\mathrm{b}):&\quad X^{\bullet 2} = X, Y^{\bullet 2} = 0 \text{ and } X \bullet Y = Y/2,\\
3\ (\mathrm{a}):&\quad X^{\bullet 2} = Y,\ Y^{\bullet 2} = 0 \text{ and } X \bullet Y = 0,\\
(\mathrm{b}):&\quad X^{\bullet 2} = 0,\ Y^{\bullet 2} = 0 \text{ and } X \bullet Y = 0.
\end{aligned}
$$

How to get all orbits isomorphic to 1(a)?

(1) Apply a congruence: we get $U = \mathbf{1}_n$

(2) Apply an orthogonal congruence: we get $X = \mathrm{Diag}(\mathbf{1}_k, \mathbf{0}_{n-k})$

(3) Now we see that $Y = \mathrm{Diag}(\mathbf{0}_k, Z)$ with $Z^2 = Z$

(4) Apply an orthogonal congruence: we get $Z = \mathrm{Diag}(\mathbf{1}_\ell, \mathbf{0}_{n-k-\ell})$

Jordan nets of symmetric 4×4 matrices

codim 9

$$\begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}$$

codim 10

$$\begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix} \quad \begin{pmatrix} x & & & \\ & z & & y \\ & & y & \\ & y & & \end{pmatrix} \quad \begin{pmatrix} x & y & & \\ y & z & & \\ & & x & y \\ & & y & z \end{pmatrix}$$

codim 11

$$\begin{pmatrix} z & y & x & \\ y & x & & \\ x & & & \\ & & & x \end{pmatrix} \quad \begin{pmatrix} y & & & x \\ & z & x & \\ & x & & \\ x & & & \end{pmatrix} \quad \begin{pmatrix} & x & z & \\ x & & & \\ z & & & y \\ & & y & \end{pmatrix}$$

codim 12

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$$\begin{aligned}
1\ (\mathrm{a}):&\ X^{\bullet 2} = X, Y^{\bullet 2} = Y \text{ and } X \bullet Y = 0,\\
(\mathrm{b}):&\ X^{\bullet 2} = U,\ Y^{\bullet 2} = U \text{ and } X \bullet Y = 0,\\
2\ (\mathrm{a}):&\ X^{\bullet 2} = X, Y^{\bullet 2} = 0 \text{ and } X \bullet Y = 0,\\
(\mathrm{b}):&\ X^{\bullet 2} = X, Y^{\bullet 2} = 0 \text{ and } X \bullet Y = Y/2,\\
3\ (\mathrm{a}):&\ X^{\bullet 2} = Y,\ Y^{\bullet 2} = 0 \text{ and } X \bullet Y = 0,\\
(\mathrm{b}):&\ X^{\bullet 2} = 0,\ Y^{\bullet 2} = 0 \text{ and } X \bullet Y = 0.
\end{aligned}$$

Question - orbits of type 3(b)

(1) Apply a congruence: we get $U = \mathbf{1}_n$

(2) Now we see that $X^2 = Y^2 = XY + YX = 0$

(3) Can we classify $\mathbb{C}\{X, Y\}$ up to orthogonal congruence?

Degenerating orbits

Go to nets of quadrics: $\mathcal{L} \rightsquigarrow (a,b,c,d)\,\mathcal{L}(a,b,c,d)^\top \subseteq \mathbb{C}[a,b,c,d]_2$

$$\mathrm{Diag}(x\mathbf{1}_2, y, z) \rightsquigarrow \mathrm{span}(a^2+b^2, c^2, d^2)$$

The group $\mathrm{GL}(4)$ now acts using coordinate transformations.

The orbit of $\mathrm{span}(a^2+b^2, c^2, d^2)$ contains

$$\mathrm{span}(a^2+b^2, (d+tc)^2, d^2) = \mathrm{span}(a^2+b^2, 2cd+tc^2, d^2)$$

for all $t \neq 0$. Letting $t \to 0$, we get

$$\mathrm{span}(a^2+b^2, 2cd, d^2) \leftrightsquigarrow \begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix}$$

Jordan nets of symmetric 4×4 matrices

codim 9

$$\begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}$$

codim 10

$$\begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix} \quad \begin{pmatrix} x & & & \\ & z & & y \\ & & y & \\ & y & & \end{pmatrix} \quad \begin{pmatrix} x & y & & \\ y & z & & \\ & & x & y \\ & & y & z \end{pmatrix}$$

codim 11

$$\begin{pmatrix} z & y & x & \\ y & x & & \\ x & & & \\ & & & x \end{pmatrix} \quad \begin{pmatrix} y & & & x \\ & z & x & \\ & x & & \\ x & & & \end{pmatrix} \quad \begin{pmatrix} & x & z & \\ x & & & \\ z & & & y \\ & & y & \end{pmatrix}$$

codim 12

$$\begin{pmatrix} z & y & & x \\ y & & x & \\ & x & & \\ x & & & \end{pmatrix}$$

Jordan nets of symmetric 4×4 matrices

$$\min\{\operatorname{rk}(X) \mid X \in \mathcal{L} \setminus \{0\}\} \leq 1$$

codim 9

$$\begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}$$

codim 10

$$\begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix} \quad \begin{pmatrix} x & & & \\ & z & & y \\ & & y & \\ & y & & \end{pmatrix} \quad \begin{pmatrix} x & y & & \\ y & z & & \\ & & x & y \\ & & y & z \end{pmatrix}$$

codim 11

$$\begin{pmatrix} z & y & x & \\ y & x & & \\ x & & & \\ & & & x \end{pmatrix} \quad \begin{pmatrix} y & & & x \\ & z & x & \\ & x & & \\ x & & & \end{pmatrix} \quad \begin{pmatrix} & x & z & \\ x & & & \\ z & & & y \\ & & y & \end{pmatrix}$$

codim 12

$$\begin{pmatrix} z & y & & x \\ y & & x & \\ & x & & \\ x & & & \end{pmatrix}$$

Jordan nets of symmetric 4×4 matrices

Proposition

$\begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}$ does not degenerate to $\begin{pmatrix} & x & z & \\ x & & & \\ z & & & y \\ & & y & \end{pmatrix}$.

Proof.

The closed set

$$\{(\mathcal{L}, X) \in \mathrm{Gr}(m, \mathbb{S}^n) \times \mathbb{P}(\mathbb{S}^n) \mid X \in \mathcal{L}, \mathrm{rk}(X) \leq 1\}$$

projects to $\mathrm{Gr}(m, \mathbb{S}^n)$ along the complete variety $\mathbb{P}(\mathbb{S}^n)$.

This projection is therefore closed. It consists of all spaces $\mathcal{L}$ where

$$\min\{\mathrm{rk}(X) \mid X \in \mathcal{L} \setminus \{0\}\} \leq 1$$

This condition holds for the space on the left. So also for all its degenerations. And, it does not hold for the space on the right. □

Jordan nets of symmetric 4×4 matrices

contains a pencil $\mathcal{P} \in \mathrm{Gr}(2, \mathbb{S}^4)$
with $\det(\mathcal{P}) = 0$ and
$\det(Q\,\mathcal{P}\,Q^\top) = \square$
for all $Q \in \mathbb{C}^{2\times 4}$

codim 9

$$\begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}$$

codim 10

$$\begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix} \quad \begin{pmatrix} x & & & \\ & z & & y \\ & & y & \\ & y & & \end{pmatrix} \quad \begin{pmatrix} x & y & & \\ y & z & & \\ & & x & y \\ & & y & z \end{pmatrix}$$

codim 11

$$\begin{pmatrix} z & y & x & \\ y & x & & \\ x & & & \\ & & & x \end{pmatrix} \quad \begin{pmatrix} y & & & x \\ & z & x & \\ & x & & \\ x & & & \end{pmatrix} \quad \begin{pmatrix} & x & z & \\ x & & & \\ z & & & y \\ & & y & \end{pmatrix}$$

codim 12

$$\begin{pmatrix} z & y & & x \\ y & & x & \\ & x & & \\ x & & & \end{pmatrix}$$

Proposition

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Proof.

The closed subset

$$\left\{ (\mathcal{L}, \mathcal{P}) \in \mathrm{Gr}(3, \mathbb{S}^n) \times \mathrm{Gr}(2, \mathbb{S}^n) \;\middle|\; \begin{array}{c} \mathcal{P} \subseteq \mathcal{L}, \;\; \det(\mathcal{P}) = 0, \\ \forall Q \in \mathbb{C}^{2\times 4} : \det(Q\,\mathcal{P}\,Q^\top) = \square \end{array} \right\}$$

projects to a closed subset of $\mathrm{Gr}(m, \mathbb{S}^n)$.

The orbit of the space on the left is in this subset. The space on the right is not. $\square$

Jordan nets of symmetric 4×4 matrices

codim 9

$$\begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}$$

general minimum polynomial has degree ≤ 2

codim 10

$$\begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix} \quad \begin{pmatrix} x & & & \\ & z & & y \\ & & y & \\ & y & & \end{pmatrix} \quad \begin{pmatrix} x & y & & \\ y & z & & \\ & & x & y \\ & & y & z \end{pmatrix}$$

codim 11

$$\begin{pmatrix} z & y & x & \\ y & x & & \\ x & & & \\ & & & x \end{pmatrix} \quad \begin{pmatrix} y & & & x \\ & z & x & \\ & x & & \\ x & & & \end{pmatrix} \quad \begin{pmatrix} & x & z & \\ x & & & \\ z & & & y \\ & & y & \end{pmatrix}$$

codim 12

$$\begin{pmatrix} z & y & & x \\ y & & x & \\ & x & & \\ x & & & \end{pmatrix}$$

Proposition

$\begin{pmatrix} x & y & & \\ y & z & & \\ & & x & y \\ & & y & z \end{pmatrix}$ does not degenerate to $\begin{pmatrix} z & y & x & \\ y & x & & \\ x & & & \\ & & & x \end{pmatrix}$;

Proof.

Let X, Y, Z be a basis of $\mathcal{L}$ and consider the following condition: For all $x, y, z \in \mathbb{C}$ and all $U \in \mathcal{L}_{\text{inv}}$,

$$U, \quad W, \quad W \bullet_U W$$

are linearly dependent for $W = xX + yY + zZ$.

This condition is closed, is satisfied by the orbit of the space on the left and not satisfied by the space on the right. □

Jordan nets of symmetric 4×4 matrices

determinant of form fg^3 with f, g linear

codim 9

$$\begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}$$

codim 10

$$\begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix} \quad \begin{pmatrix} x & & & \\ & z & & y \\ & & y & \\ & y & & \end{pmatrix} \quad \begin{pmatrix} x & y & & \\ y & z & & \\ & & x & y \\ & & y & z \end{pmatrix}$$

codim 11

$$\begin{pmatrix} z & y & x & \\ y & x & & \\ x & & & \\ & & & x \end{pmatrix} \quad \begin{pmatrix} y & & & x \\ & z & x & \\ & x & & \\ x & & & \end{pmatrix} \quad \begin{pmatrix} & x & z & \\ x & & & \\ z & & & y \\ & & y & \end{pmatrix}$$

codim 12

$$\begin{pmatrix} z & y & & x \\ y & & x & \\ & x & & \\ x & & & \end{pmatrix}$$

Proposition

The condition "determinant of form fg^3 with f, g linear" is closed?

Proof.

The condition states that

$$(X, Y, Z) \mapsto \det(xX + yY + zZ) \in \mathbb{C}[x, y, z]_4$$

maps a basis X, Y, Z of $\mathcal{L}$ into $\{fg^3 \mid f, g \in \mathbb{C}[x, y, z]_1\}$.

This set is (the cone of) the image of the map

$$\begin{array}{rcl} \mathbb{P}(\mathbb{C}[x, y, z]_1) \times \mathbb{P}(\mathbb{C}[x, y, z]_1) & \to & \mathbb{P}(\mathbb{C}[x, y, z]_4) \\ ([f], [g]) & \mapsto & [fg^3] \end{array}$$

and hence closed. □

Future directions

(1) Study m-dimensional subspaces of $\mathbb{S}^n$ for other (m, n).

- Classification of Jordan nets in $\mathbb{S}^n$.
- Finding all the degenerations.
- Are (variations of) the closed conditions we looked at enough to show that these degenerations are the only ones?

(2) Study nonregular subspaces (pencils) $\mathcal{L}$, i.e. where $\det(\mathcal{L}) = 0$.

Thank you for your attention!

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