

# Jordan Algebras of Symmetric Matrices

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Let  $\mathbb{S}^n$  be the space of symmetric  $n \times n$  matrices over  $\mathbb{C}$ .

The Grassmannian  $\text{Gr}(m, \mathbb{S}^n)$  consists of  $m$ -dimensional  $\mathcal{L} \subseteq \mathbb{S}^n$ .

We here consider *regular* subspaces  $\mathcal{L} \subseteq \mathbb{S}^n$ :

$$\mathcal{L}_{\text{inv}} := \{X \in \mathcal{L} \mid \det(X) \neq 0\} \neq \emptyset$$

## Definition

The *reciprocal variety*  $\mathcal{L}^{-1}$  is  $\overline{\{X^{-1} \mid X \in \mathcal{L}_{\text{inv}}\}} \subseteq \mathbb{S}^n$ .

## Goal

Understand the  $\mathcal{L}$  where the variety  $\mathcal{L}^{-1}$  is a linear space in  $\mathbb{S}^n$ .

The motivation for this goal arises in optimization (semidefinite programming) and in statistics (Gaussian models that are linear in covariance matrices and concentration matrices).



## Examples

$$\begin{pmatrix} x & y & & w \\ y & z & -w & \\ & -w & x & y \\ w & & y & z \end{pmatrix}, \quad \begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & y \\ & & & & z \end{pmatrix}, \quad \begin{pmatrix} x & & & & \\ & z & & & \\ & & z & y & \\ & & y & & \\ & y & & & y \end{pmatrix},$$

$$\begin{pmatrix} z & y & x & & & \\ y & x & & & & \\ x & & & & & \\ & & & y & x & \\ & & & x & & \\ & & & & & x \end{pmatrix}, \quad \begin{pmatrix} & y & z & & x \\ y & & & & \\ z & & & x & \\ & & x & & \\ & x & & & \\ x & & & & \end{pmatrix}$$



## Theorem (B-Eisenmann-Sturmfels 2020, Jensen 1988)

For  $\mathcal{L} \in \text{Gr}(m, \mathbb{S}^n)$  and  $U \in \mathcal{L}_{\text{inv}}$ , the following are equivalent:

- (a) The reciprocal variety  $\mathcal{L}^{-1}$  is also a linear space in  $\mathbb{S}^n$ .
- (b)  $\mathcal{L}$  is a subalgebra of the Jordan algebra  $(\mathbb{S}^n, \bullet_U)$ .
- (c)  $\mathcal{L}^{-1}$  equals  $\mathcal{L}$  up to congruence; namely  $\mathcal{L}^{-1} = U^{-1} \mathcal{L} U^{-1}$ .

We say that  $\mathcal{L}$  is a *Jordan space* when these equivalent conditions are satisfied.

## Definition

For  $U \in \mathbb{S}_{\text{inv}}^n$ , we define an algebra structure on  $\mathbb{S}^n$  by

$$X \bullet_U Y := (XU^{-1}Y + YU^{-1}X)/2$$

for all  $X, Y \in \mathbb{S}^n$ . This makes  $\mathbb{S}^n$  into a (unital) Jordan algebra:

$$X^{\bullet 2} \bullet (X \bullet Y) = X \bullet (X^{\bullet 2} \bullet Y).$$



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## Remark 1

For congruent subspaces  $\mathcal{L}$  and  $\mathcal{L}' = P \mathcal{L} P^\top$ :

$\mathcal{L}$  is a Jordan space  $\Leftrightarrow \mathcal{L}'$  is also a Jordan space

## Remark 2

All choices of unit  $U$  lead to isomorphic Jordan algebras  $(\mathcal{L}, \bullet_U)$ .



Let  $V$  be a vector space.

## Definition

The projective space

$$\mathbb{P}(V) := \{[v] \mid v \in V \setminus \{0\}\}$$

where  $[v] = [w]$  when  $v = \lambda w$  for some  $\lambda \neq 0$ .

A subvariety of  $\mathbb{P}(V)$  is defined by homogeneous polynomials:

$$\text{for some } d \geq 0: \quad f(\lambda v) = \lambda^d f(v) \text{ for all } \lambda \in \mathbb{C} \text{ and } v \in V$$

## Theorem

Projective spaces are complete. So projections of closed subsets  $Y \subseteq X \times \mathbb{P}(V)$  to  $X$  are closed. In particular, images from projective spaces are closed.



## Example

The image of the map

$$\begin{aligned}\mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^n) &\rightarrow \mathbb{P}(\mathbb{C}^n \times \mathbb{C}^n) \\ ([\lambda, \mu], [v]) &\mapsto [\lambda v, \mu v]\end{aligned}$$

is closed.

In fact, it is the set of linearly dependent vectors in  $\mathbb{C}^n \times \mathbb{C}^n$ .

So a point  $[(x_1, \dots, x_n), (y_1, \dots, y_n)]$  is in the image of the map if and only if the matrix

$$\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$$

has rank  $\leq 1$ . The polynomials  $x_i y_j - x_j y_i$  are homogeneous.

# The variety of Jordan spaces



Let  $V$  be a vector space of dimension  $n$ .

## Coordinate systems for the Grassmannian

A subspace  $\mathcal{L} \in \text{Gr}(m, V)$  can be represented by:

	primal	dual
Stiefel	$(H_1, \dots, H_{n-m}) \in (V^*)^{n-m}$	$(X_1, \dots, X_m) \in V^m$
Plücker	$H_1 \wedge \dots \wedge H_{n-m} \in \wedge^{n-m} V^*$	$X_1 \wedge \dots \wedge X_m \in \wedge^m V$

Here

$$\mathcal{L} = \{v \in V \mid H_1(v), \dots, H_{n-m}(v) = 0\} = \text{span}(X_1, \dots, X_m)$$

## Proposition

The subset  $\text{Jo}(m, \mathbb{S}^n)$  consisting of all Jordan spaces  $\mathcal{L}$  is a subvariety of  $\text{Gr}(m, \mathbb{S}^n)$ .





## Proof

The subspace  $\mathcal{L} \in \text{Gr}(m, \mathbb{S}^n)$  is a Jordan space  $\Leftrightarrow$

(b)  $\mathcal{L}$  is a subalgebra of the Jordan algebra  $(\mathbb{S}^n, \bullet_U)$ .

for all  $U \in \mathcal{L}_{\text{inv}}$ .

Let  $X_1, \dots, X_m$  be a basis of  $\mathcal{L}$ . (Dual Stiefel coordinates)

Then (b) for all  $U \in \mathcal{L}_{\text{inv}} \Leftrightarrow$

$$X_1, \dots, X_m, X_i \bullet_U X_j$$

are linearly dependent for all  $1 \leq i \leq j \leq m$  and  $U \in \mathcal{L}_{\text{inv}}$



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Let  $X_1, \dots, X_m$  be a basis of  $\mathcal{L}$ . (Dual Stiefel coordinates)

Then (b) for all  $U \in \mathcal{L}_{\text{inv}} \Leftrightarrow$

$$X_1, \dots, X_m, (X_i U^{-1} X_j + X_j U^{-1} X_i)/2$$

are linearly dependent for all  $1 \leq i \leq j \leq m$  and  $U \in \mathcal{L}_{\text{inv}}$



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Then (b) for all  $U \in \mathcal{L}_{\text{inv}} \Leftrightarrow$

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$$X_1, \dots, X_m, (X_i \text{adj}(U)X_j + X_j \text{adj}(U)X_i)$$

are linearly dependent for all  $1 \leq i \leq j \leq m$  and

$$U = c_1 X_1 + \dots + c_m X_m \in \mathcal{L}$$

for all  $c_1, \dots, c_m \in \mathbb{C}$ .





We call elements of  $\text{Gr}(2, \mathbb{S}^n)$  *pencils* of symmetric matrices.

Congruence orbits of regular pencils are classified by Segre symbols.

## Definition

Let  $\mathcal{L} \in \text{Gr}(2, \mathbb{S}^n)$  be a regular pencil. The Segre symbol  $\sigma$  of  $\mathcal{L}$  is a multiset of partitions adding up to  $n$ .

Pick a basis  $X, Y$  of  $\mathcal{L}$  with  $Y \in \mathcal{L}_{\text{inv}}$ . Then the Segre symbol of  $\mathcal{L}$  is given by sizes of Jordan blocks of  $XY^{-1}$ .

## Examples

$$\begin{pmatrix} x & & & & \\ & x & & & \\ & & y & & \\ & & & y & \\ & & & & y \end{pmatrix} \text{ and } \begin{pmatrix} y & x & & & \\ x & & & & \\ & & y & & \\ & & & y & \\ & & & & y \end{pmatrix}$$

have Segre symbols  $[(1, 1), (1, 1, 1)]$  and  $[(2), (1, 1, 1)]$ .



We know which Segre symbols correspond to Jordan pencils.

### Theorem (Fevola-Mandelstam-Sturmfels 2020)

A pencil is a Jordan space exactly when its Jordan symbol is of the form  $\sigma = [(1, \dots, 1), (1, \dots, 1)]$  or  $\sigma = [(2, \dots, 2, 1, \dots, 1)]$ .

$$\begin{pmatrix} x & & & & & \\ & \ddots & & & & \\ & & x & & & \\ & & & y & & \\ & & & & \ddots & \\ & & & & & y \end{pmatrix} \quad \begin{pmatrix} y & x & & & & \\ x & & & & & \\ & & \ddots & & & \\ & & & y & x & \\ & & & x & & \\ & & & & & x \\ & & & & & & \ddots & \\ & & & & & & & x \end{pmatrix}$$

The irreducible components of  $\text{Jo}(2, \mathbb{S}^n)$  are the orbits closures of the diagonalizable pencils.



We call elements of  $\text{Gr}(3, \mathbb{S}^n)$  *nets* of symmetric matrices.

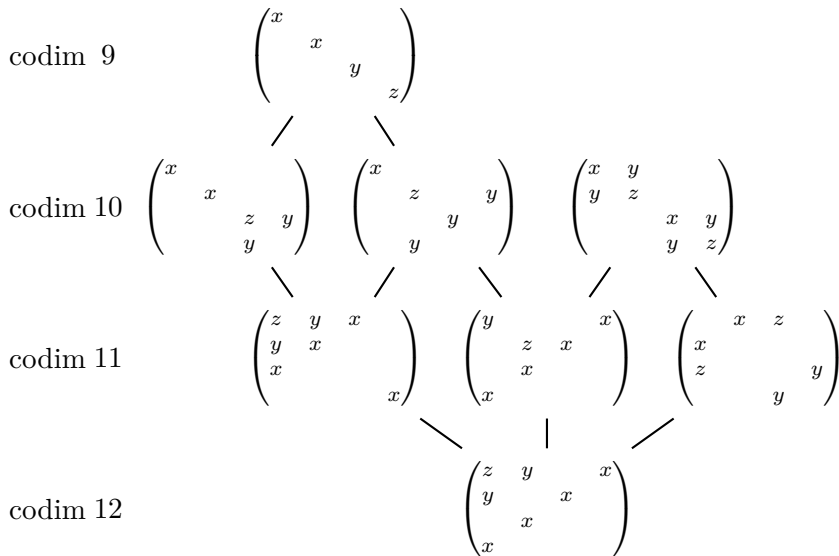
For  $n = 2$ , we have  $\text{Gr}(3, \mathbb{S}^2) = \{\mathbb{S}^2\}$  and  $\mathbb{S}^2$  is a Jordan net.

For  $n = 3$ , all Jordan nets are congruent to one of:

$$\begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \rightarrow \begin{pmatrix} x & & \\ & z & y \\ & y & \end{pmatrix} \rightarrow \begin{pmatrix} z & y & x \\ y & x & \\ x & & \end{pmatrix}$$

For  $n = 4$ , the diagram becomes more interesting.

# Jordan nets of symmetric $4 \times 4$ matrices







## Theorem

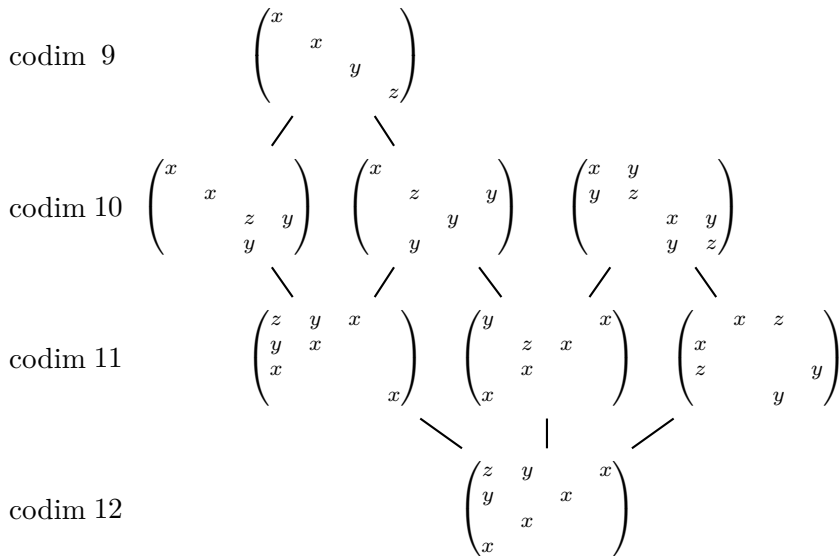
Every (unital) Jordan algebra  $\mathcal{A}$  of dimension 3 over  $\mathbb{C}$  is isomorphic to the Jordan algebra  $\mathbb{C}\{U, X, Y\}$  with unit  $U$ , where the product is given by

- 1 (a):  $X \bullet^2 = X, Y \bullet^2 = Y$  and  $X \bullet Y = 0$ ,  
   (b):  $X \bullet^2 = U, Y \bullet^2 = U$  and  $X \bullet Y = 0$ ,
- 2 (a):  $X \bullet^2 = X, Y \bullet^2 = 0$  and  $X \bullet Y = 0$ ,  
   (b):  $X \bullet^2 = X, Y \bullet^2 = 0$  and  $X \bullet Y = Y/2$ ,
- 3 (a):  $X \bullet^2 = Y, Y \bullet^2 = 0$  and  $X \bullet Y = 0$ ,  
   (b):  $X \bullet^2 = 0, Y \bullet^2 = 0$  and  $X \bullet Y = 0$ .

## How to get all orbits isomorphic to 1(a)?

- (1) Apply a congruence: we get  $U = \mathbf{1}_n$
- (2) Apply an orthogonal congruence: we get  $X = \text{Diag}(\mathbf{1}_k, \mathbf{0}_{n-k})$
- (3) Now we see that  $Y = \text{Diag}(\mathbf{0}_k, Z)$  with  $Z^2 = Z$
- (4) Apply an orthogonal congruence: we get  $Z = \text{Diag}(\mathbf{1}_\ell, \mathbf{0}_{n-k-\ell})$

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## Theorem

Every (unital) Jordan algebra  $\mathcal{A}$  of dimension 3 over  $\mathbb{C}$  is isomorphic to the Jordan algebra  $\mathbb{C}\{U, X, Y\}$  with unit  $U$ , where the product is given by

- 1 (a):  $X^{\bullet 2} = X, Y^{\bullet 2} = Y$  and  $X \bullet Y = 0$ ,  
(b):  $X^{\bullet 2} = U, Y^{\bullet 2} = U$  and  $X \bullet Y = 0$ ,
- 2 (a):  $X^{\bullet 2} = X, Y^{\bullet 2} = 0$  and  $X \bullet Y = 0$ ,  
(b):  $X^{\bullet 2} = X, Y^{\bullet 2} = 0$  and  $X \bullet Y = Y/2$ ,
- 3 (a):  $X^{\bullet 2} = Y, Y^{\bullet 2} = 0$  and  $X \bullet Y = 0$ ,  
(b):  $X^{\bullet 2} = 0, Y^{\bullet 2} = 0$  and  $X \bullet Y = 0$ .

## Question - orbits of type 3(b)

- (1) Apply a congruence: we get  $U = \mathbf{1}_n$
- (2) Now we see that  $X^2 = Y^2 = XY + YX = 0$
- (3) Can we classify  $\mathbb{C}\{X, Y\}$  up to orthogonal congruence?



## Degenerating orbits

Go to nets of quadrics:  $\mathcal{L} \rightsquigarrow (a, b, c, d) \mathcal{L}(a, b, c, d)^\top \subseteq \mathbb{C}[a, b, c, d]_2$

$$\text{Diag}(x\mathbf{1}_2, y, z) \rightsquigarrow \text{span}(a^2 + b^2, c^2, d^2)$$

The group  $\text{GL}(4)$  now acts using coordinate transformations.

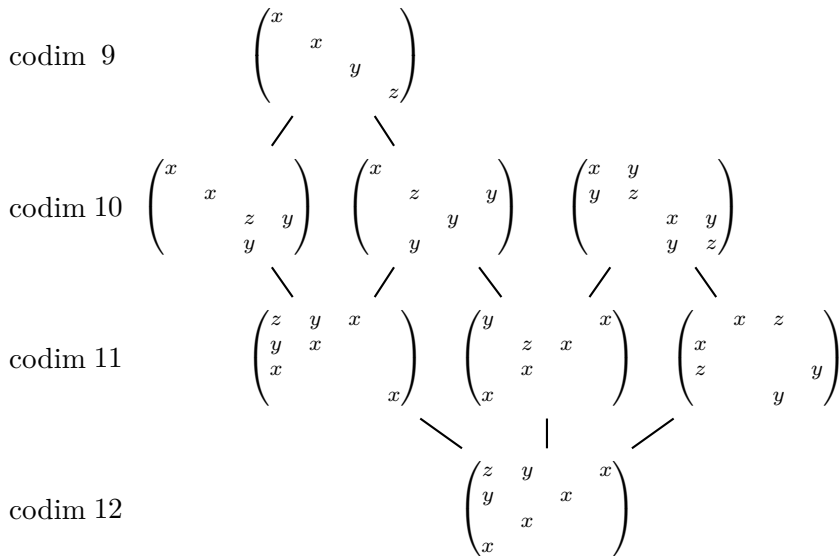
The orbit of  $\text{span}(a^2 + b^2, c^2, d^2)$  contains

$$\text{span}(a^2 + b^2, (d + tc)^2, d^2) = \text{span}(a^2 + b^2, 2cd + tc^2, d^2)$$

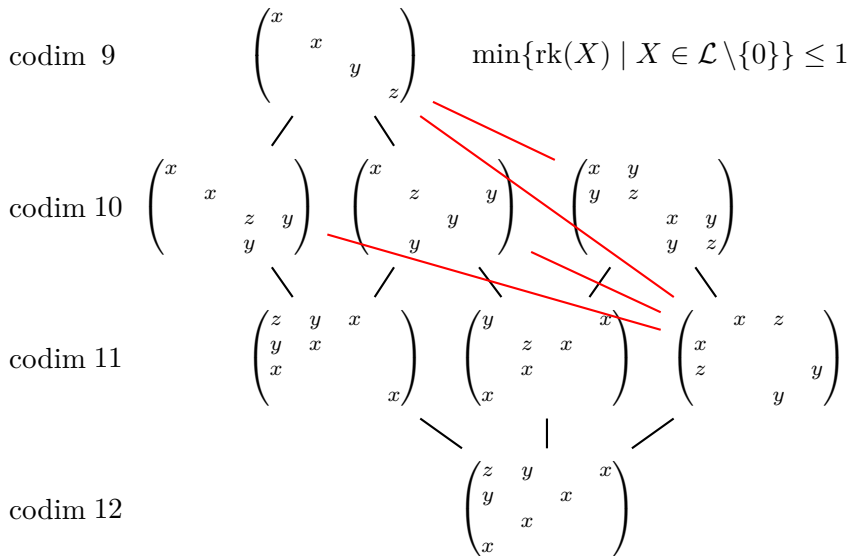
for all  $t \neq 0$ . Letting  $t \rightarrow 0$ , we get

$$\text{span}(a^2 + b^2, 2cd, d^2) \leftarrow \begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix}$$

# Jordan nets of symmetric $4 \times 4$ matrices



# Jordan nets of symmetric $4 \times 4$ matrices





## Proposition

$$\begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix} \text{ does not degenerate to } \begin{pmatrix} x & & x & z \\ & & & \\ z & & & \\ & & & y \end{pmatrix}.$$

## Proof.

The closed set

$$\{(\mathcal{L}, X) \in \text{Gr}(m, \mathbb{S}^n) \times \mathbb{P}(\mathbb{S}^n) \mid X \in \mathcal{L}, \text{rk}(X) \leq 1\}$$

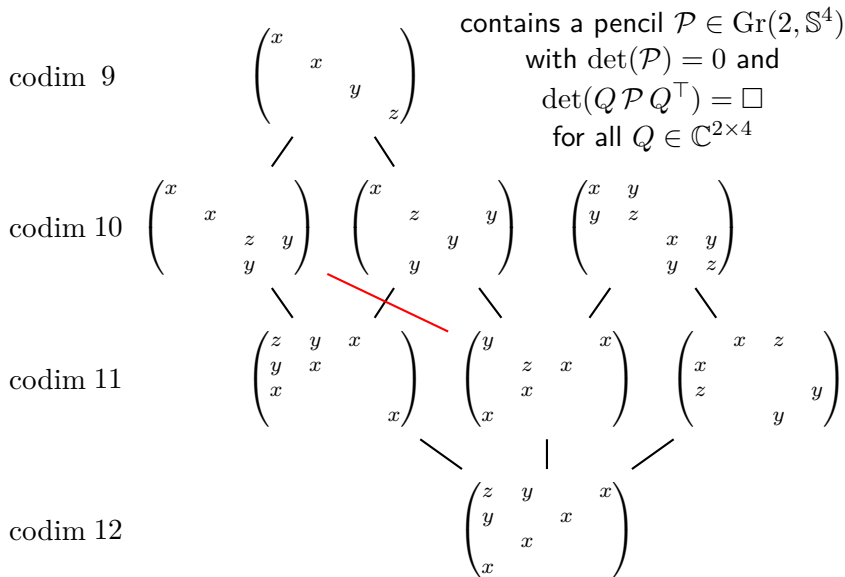
projects to  $\text{Gr}(m, \mathbb{S}^n)$  along the complete variety  $\mathbb{P}(\mathbb{S}^n)$ .

This projection is therefore closed. It consists of all spaces  $\mathcal{L}$  where

$$\min\{\text{rk}(X) \mid X \in \mathcal{L} \setminus \{0\}\} \leq 1$$

This condition holds for the space on the left. So also for all its degenerations. And, it does not hold for the space on the right.  $\square$

# Jordan nets of symmetric $4 \times 4$ matrices







## Proposition

$$\begin{pmatrix} x & & & \\ & x & & \\ & & z & y \\ & & y & \end{pmatrix} \text{ does not degenerate to } \begin{pmatrix} y & & & x \\ & z & x & \\ & x & & \\ x & & & \end{pmatrix}.$$

## Proof.

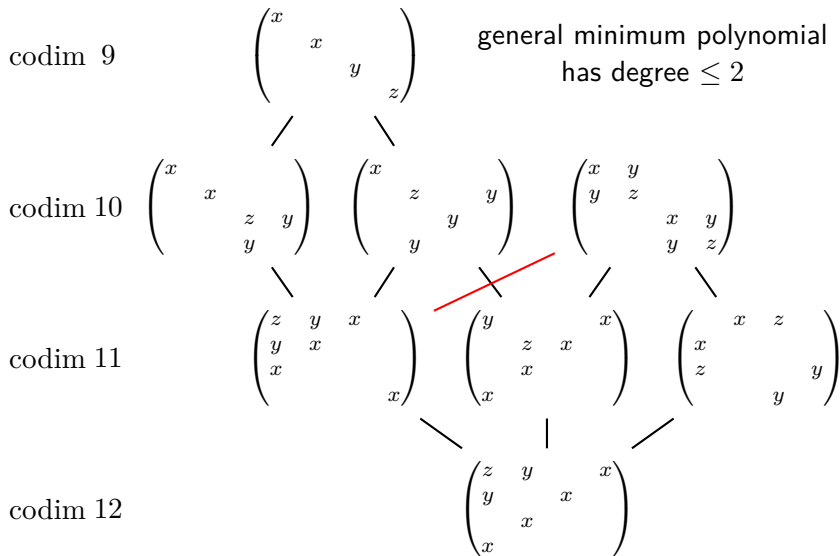
The closed subset

$$\left\{ (\mathcal{L}, \mathcal{P}) \in \text{Gr}(3, \mathbb{S}^n) \times \text{Gr}(2, \mathbb{S}^n) \mid \begin{array}{l} \mathcal{P} \subseteq \mathcal{L}, \det(\mathcal{P}) = 0, \\ \forall Q \in \mathbb{C}^{2 \times 4} : \det(Q \mathcal{P} Q^T) = 0 \end{array} \right\}$$

projects to a closed subset of  $\text{Gr}(m, \mathbb{S}^n)$ .

The orbit of the space on the left is in this subset. The space on the right is not. □

# Jordan nets of symmetric $4 \times 4$ matrices





## Proposition

$\begin{pmatrix} x & y & & \\ y & z & & \\ & & x & y \\ & & y & z \end{pmatrix}$  does not degenerate to  $\begin{pmatrix} z & y & x & \\ y & x & & \\ x & & & \\ & & & x \end{pmatrix}$ ;

## Proof.

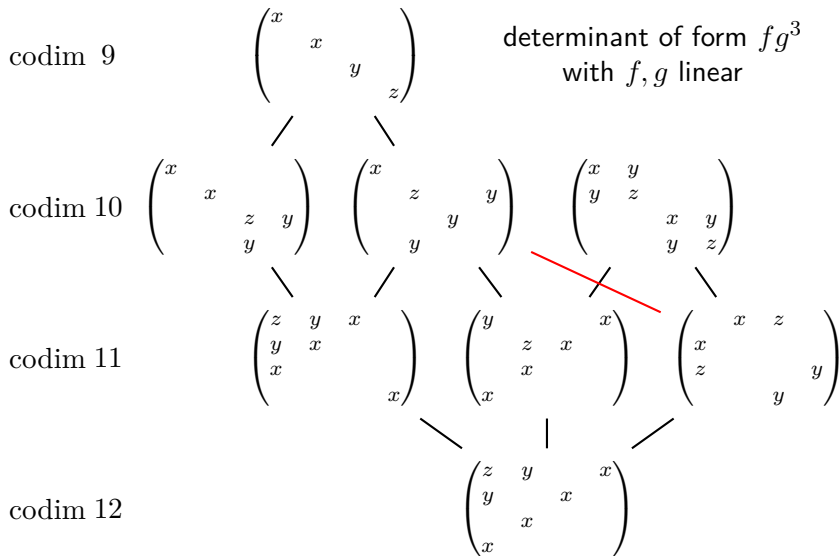
Let  $X, Y, Z$  be a basis of  $\mathcal{L}$  and consider the following condition:  
For all  $x, y, z \in \mathbb{C}$  and all  $U \in \mathcal{L}_{\text{inv}}$ ,

$$U, W, W \bullet_U W$$

are linearly dependent for  $W = xX + yY + zZ$ .

This condition is closed, is satisfied by the orbit of the space on the left and not satisfied by the space on the right.  $\square$

# Jordan nets of symmetric $4 \times 4$ matrices





## Proposition

The condition "determinant of form  $fg^3$  with  $f, g$  linear" is closed?

## Proof.

The condition states that

$$(X, Y, Z) \mapsto \det(xX + yY + zZ) \in \mathbb{C}[x, y, z]_4$$

maps a basis  $X, Y, Z$  of  $\mathcal{L}$  into  $\{fg^3 \mid f, g \in \mathbb{C}[x, y, z]_1\}$ .

This set is (the cone of) the image of the map

$$\begin{aligned} \mathbb{P}(\mathbb{C}[x, y, z]_1) \times \mathbb{P}(\mathbb{C}[x, y, z]_1) &\rightarrow \mathbb{P}(\mathbb{C}[x, y, z]_4) \\ ([f], [g]) &\mapsto [fg^3] \end{aligned}$$

and hence closed. □



- (1) Study  $m$ -dimensional subspaces of  $\mathbb{S}^n$  for other  $(m, n)$ .
  - Classification of Jordan nets in  $\mathbb{S}^n$ .
  - Finding all the degenerations.
  - Are (variations of) the closed conditions we looked at enough to show that these degenerations are the only ones?
- (2) Study nonregular subspaces (pencils)  $\mathcal{L}$ , i.e. where  $\det(\mathcal{L}) = 0$ .

**Thank you for your attention!**



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*Jordan Algebras of Symmetric Matrices*

preprint



Claudia Fevola, Yelena Mandelshtam, Bernd Sturmfels

*Pencils of Quadrics: Old and New*

preprint